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RAY TRANSFORM
ON RIEMANNIAN MANIFOLDS

Eight lectures on integral geometry

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Preface

What is integral geometry? Since the famous paper by I. Radon in 1917, it has been agreed that integral geometry problems consist in determining some function or a more general object (cohomology class, tensor field, etc.) on a manifold, given its integrals over submanifolds of a prescribed class. In these lectures we only consider integral geometry problems for which the above-mentioned submanifolds are one-dimensional. Strictly speaking, the latter are always geodesics of a fixed Riemannian metric, in particular straight lines in Euclidean space. The exception is Lecture 1 in which we consider an arbitrary regular family of curves in a two-dimensional domain.

Stimulated by intrinsic demands of mathematics, in recent years integral geometry has gain a powerful impetus from computer tomography. Now integral geometry serves as a mathematical background for tomography which in turn provides most of the problems for the former.

The most part of the lectures deals with integral geometry of symmetric tensor fields. This branch of integral geometry can be viewed as a mathematical basis for tomography of anisotropic media whose interaction with sounding radiation depends essentially on the direction in which the latter propagates.

The lectures were first delivered in the University of Washington in May of 1999. The corresponding notes can be found on the web cite **math.washington.edu/~sharafut/Ray_transform.dvi**. Comparing with these notes, the present lectures contain the following additions and improvements.

1. In the previous notes as well as in my book [77], the main hypothesis of Theorem 3.4.3 looked as follows: $k^+(M, g) < 1/(m + 1)$. Now this inequality is replaced with the following one: $k^+(M, g) < (m + 2n - 1)/m(m + n)$. This implies the corresponding improvement of Theorem 5.1.1. I have noticed the possibility of this improvement just when giving a lecture in the University of Washington.

2. In the previous notes, Theorem 8.1.4 had the very unpleasant hypothesis on the W_p^1-regularity of the stable and unstable distributions. As my colleague Nurlan Dairbekov has noticed, the hypothesis can be omitted. The theorem is stronger now, and the proof is simpler.

3. Two new sections, 4.4 and 8.10, are included. They contain recent results on integral geometry of surfaces without focal points which are obtained in a joint work of the author and Gunther Uhlmann.

Contents

Lecture 1
Inverse kinematic problem
of seismics on plane
(Instead of introduction)

Here we present two results by R. G. Mukhometov [55, 56]. His proofs are very elementary, i.e., they need no preliminary knowledge. At the same time these papers contain the most general results on two-dimensional integral geometry that are known till now. Therefore I have chosen them as a good introduction to integral geometry.

1.1 The linear two-dimensional problem of integral geometry

1.1.1 Posing the problem and formulating the result

Let D be a bounded simply connected domain on the plane whose boundary is a C^1-smooth closed curve $\delta = \partial D$. We parameterize δ by the arc length:

$$x = \delta^1(t), \ y = \delta^2(t) \quad (0 \le t \le T)$$

where T is the length of δ.

Let a two-parametric family Γ of curves be given in $\bar{D}$ which satisfies the following conditions (that mean the family Γ is of the same qualitative behavior as the family of straight segments in a disk):

(i) Every two different points of $\bar{D}$ are joint by a unique curve of the family Γ.

(ii) The endpoints of a curve $\gamma \in \Gamma$ belong to δ, inner points of γ belong to D, the lengths of all curves $\gamma \in \Gamma$ are uniformly bounded.

(iii) For every point $(x_0, y_0) \in D$ and every direction θ, a unique curve $\gamma \in \Gamma$ passes through the point at the direction; the curve is given by the parametric equations

$$x = \gamma^1(x_0, y_0, \theta, s), \quad y = \gamma^2(x_0, y_0, \theta, s) \qquad (0 \le s \le S(x_0, y_0, \theta))$$

where s is the arc length on γ measured from (x_0, y_0), and $S(x_0, y_0, \theta)$ is the length of the segment of γ from the point (x_0, y_0) to δ.

(iv) The functions γ^1 and γ^2 belong to $C^3(G)$ with

$$G = \{(x_0, y_0, \theta, s) \mid (x_0, y_0) \in \bar{D}, \theta \in \mathbf{R}, 0 \le s \le S(x_0, y_0, \theta)\};$$

these functions are 2π-periodical in θ, and

$$\frac{1}{s} \frac{\partial(\gamma^1, \gamma^2)}{\partial(\theta, s)} \ge C > 0.$$

A family of curves satisfying these conditions is called the *regular family of curves*.

We now pose the principle problem. Let $f \in C^2(\bar{D})$, and the function

$$g(t_1, t_2) = \int_{\gamma(t_1, t_2)} f(x, y) \, ds \qquad (0 \le t_1, t_2 \le T) \tag{1.1.1}$$

be given where $\gamma(t_1, t_2)$ is the curve, of a given regular family Γ, joining the points $\delta(t_1), \delta(t_2) \in \delta$; and $ds = \sqrt{dx^2 + dy^2}$. One has to recover the function $f(x, y)$ from the known function $g(t_1, t_2)$.

Theorem 1.1.1 *Under the above-formulated condition, problem* (1.1.1) *has at most one solution* $f \in C^2(\bar{D})$ *that satisfies the stability estimate*

$$\|f\|_{L_2(D)} \leq \frac{1}{\sqrt{2\pi}} \left\| \frac{\partial g(t_1, t_2)}{\partial t_1} \right\|_{L_2([0,T] \times [0,T])}.$$

1.1.2 Proof of Theorem 1.1.1

We introduce the function

$$u(x, y, t) = \int\limits_{\tilde{\gamma}(x,y,t)} f(x, y)\, ds \qquad ((x, y) \in \bar{D};\ t \in [0, T]) \tag{1.1.2}$$

where $\tilde{\gamma}(x, y, t)$ is the segment, between the points $(x, y) \in \bar{D}$ and $(\delta^1(t), \delta^2(t))$, of the curve of the family Γ passing through these points. This function possesses the following local properties.

(1) $u \in C(\Omega)$ with $\Omega = \bar{D} \times [0, T]$.
(2) $u \in C^2(\Omega \setminus \Omega_0)$ with $\Omega_0 = \{(\delta^1(t), \delta^2(t), t)\}$.
(3) The derivatives u_x, u_y, u_t are bounded in $\Omega \setminus \Omega_0$.

These properties of the function u follow from regularity of Γ and the fact $f \in C^2(\bar{D})$. We postpone proving the properties to Section 1.1.3.

We denote by $\theta(x, y, t)$ the angle from the horizontal direction to the tangent vector of the curve $\tilde{\gamma}(x, y, t)$ at the point (x, y). Then the function u satisfies the equation

$$\cos \theta(x, y, t) u_x(x, y, t) + \sin \theta(x, y, t) u_y(x, y, t) = f(x, y). \tag{1.1.3}$$

This equation follows from the definition (1.1.2). Since the equation plays the principle role in our arguments, we present the detail proof of it.

We fix a number t and a point $(x_0, y_0) \in \bar{D}$. Parameterize the curve $\gamma(x_0, y_0, t)$ by the arc length s:

$$x = \gamma^1(s),\ y = \gamma^2(s);\quad \gamma^1(s_0) = x_0,\ \gamma^2(s_0) = y_0. \tag{1.1.4}$$

Then

$$\dot{\gamma}^1(s_0) = \cos \theta(x_0, y_0, t),\quad \dot{\gamma}^2(s_0) = \sin \theta(x_0, y_0, t),$$

and

$$u(\gamma^1(s), \gamma^2(s), t) = \int\limits_0^s f(\gamma^1(\sigma), \gamma^2(\sigma))\, d\sigma.$$

Differentiating the latter equality with respect to s, we obtain

$$u_x(\gamma^1(s), \gamma^2(s), t)\dot{\gamma}^1(s) + u_y(\gamma^1(s), \gamma^2(s), t)\dot{\gamma}^2(s) = f(\gamma^1(s), \gamma^2(s)).$$

Putting $s = s_0$ here and using (1.1.4), we arrive at the equality

$$u_x(x_0, y_0, t) \cos \theta(x_0, y_0, t) + u_y(x_0, y_0, t) \sin \theta(x_0, y_0, t) = f(x_0, y_0)$$

that coincides with (1.1.3).

The function $u(x, y, t)$ satisfies the boundary condition

$$u(x, y, t_1)|_{(x,y)=\delta(t_2)} = g(t_1, t_2). \tag{1.1.5}$$

Differentiating equation (1.1.3) with respect to t, we eliminate the function $f(x, y)$

$$Lu \equiv \frac{\partial}{\partial t}(\cos \theta u_x + \sin \theta u_y) = 0. \tag{1.1.6}$$

Now we consider system (1.1.5)–(1.1.6) as a boundary value problem for the function $u(x, y, t)$.

Our proof is based on the following differential identity.

Lemma 1.1.2 *For every function $u(x, y, t) \in C^2(D \times [0, T])$, the following identity is valid:*

$$2(-u_x \sin \theta + u_y \cos \theta) Lu = \frac{\partial \theta}{\partial t}(u_x^2 + u_y^2) + \frac{\partial}{\partial x}(u_y u_t) - \frac{\partial}{\partial y}(u_x u_t) +$$

$$+ \frac{\partial}{\partial t}\left[(-u_x \sin \theta + u_y \cos \theta)(u_x \cos \theta + u_y \sin \theta)\right].$$

Proof. The two-dimensional vector

$$(a, b) = (u_x \cos \theta + u_y \sin \theta, -u_x \sin \theta + u_y \cos \theta)$$

is the result of rotating the vector (u_x, u_y) on the angle θ. Consequently,

$$\arctan\left(\frac{u_y}{u_x}\right) = \theta + \arctan\left(\frac{b}{a}\right).$$

Differentiating this equality with respect to t, we obtain

$$\frac{u_x u_{yt} - u_y u_{xt}}{u_x^2 + u_y^2} = \theta_t + \frac{ab_t - ba_t}{a^2 + b^2}.$$

Since $a^2 + b^2 = u_x^2 + u_y^2$, the latter equality can be rewritten in the form

$$2ba_t = \theta_t(u_x^2 + u_y^2) + \frac{\partial}{\partial x}(u_y u_t) - \frac{\partial}{\partial y}(u_x u_t) + \frac{\partial}{\partial t}(ab).$$

Substituting the value of (a, b), we obtain the statement. The lemma is proved.

We are starting the proof of Theorem 1.1.1. The main idea is to apply the identity of Lemma 1.1.2 to function (1.1.2), to integrate the so obtained equality with respect to x, y, t, and then to transform the integral of divergence terms by the Gauss — Ostrogradskiǐ formula. In such the way we run into the difficulty related to singularity of the function u near the set Ω_0. Therefore we distinguish the neighborhood Ω_ε of the set Ω_0 by putting

$$\Omega_\varepsilon = \{p = (x, y, t) \in \bar{D} \times [0, T] \mid \text{dist}(p, \Omega_0) \leq \varepsilon\}$$

where dist is the distance in $\mathbf{R}^3$. We also denote by $\sigma_\varepsilon = \partial(\Omega \setminus \Omega_\varepsilon) \cap \partial \Omega_\varepsilon$ the boundary between $\Omega \setminus \Omega_\varepsilon$ and Ω_ε, and by $S_\varepsilon = \partial \Omega \setminus \partial \Omega_\varepsilon$ the rest of the boundary of Ω.

By equation (1.1.6) and Lemma 1.1.1, the following identity is valid on $D \times [0, T]$:

$$\frac{\partial \theta}{\partial t}\left(u_x^2 + u_y^2\right) = -\frac{\partial}{\partial x}(u_y u_t) + \frac{\partial}{\partial y}(u_x u_t) - \frac{\partial}{\partial t}\left[(-u_x \sin \theta + u_y \cos \theta)(u_x \cos \theta + u_y \sin \theta)\right].$$

We integrate this equality over $\Omega \setminus \Omega_\varepsilon$ and transform the right-hand integral by the Gauss — Ostrogradskiǐ formula:

$$\iiint\limits_{\Omega \setminus \Omega_\varepsilon} \frac{\partial \theta}{\partial t}(u_x^2 + u_y^2) \, dx dy dt =$$

$$= \iint\limits_{S_\varepsilon} \left[-u_y u_t \nu_x + u_x u_t \nu_y - (-u_x \sin \theta + u_y \cos \theta)(u_x \cos \theta + u_y \sin \theta)\nu_t\right] dS$$

$$+ \iint\limits_{\sigma_\varepsilon} \left[-u_y u_t \nu_x + u_x u_t \nu_y - (-u_x \sin \theta + u_y \cos \theta)(u_x \cos \theta + u_y \sin \theta)\nu_t\right] dS,$$

where $\nu = (\nu_x, \nu_y, \nu_t)$ is the unit outer normal vector to $\partial(\Omega \setminus \Omega_\varepsilon)$.

We take the limit in the latter equality as $\varepsilon \to 0$. By boundedness of the first derivatives of the function u, the last integral tends to zero; and we obtain

$$\int\limits_0^T \iint\limits_D \frac{\partial\theta}{\partial t}(u_x^2 + u_y^2)\,dxdydt =$$

$$= \iint\limits_{\partial(D\times[0,T])} \left[-u_y u_t \nu_x + u_x u_t \nu_y - (-u_x\sin\theta + u_y\cos\theta)(u_x\cos\theta + u_y\sin\theta)\nu_t\right]dS.$$

The boundary $\partial(D\times[0,T])$ consists of the three parts:

$$\partial(D\times[0,T]) = (\delta\times[0,T])\cup(D\times\{0\})\cup(D\times\{T\})\,.$$

Observe that all integrands are periodical in t with the period T, and the vector ν has the opposite values on the low and upper bottoms. Therefore integrals over $D\times\{0\}$ and $D\times\{T\}$ cancel each other. On the lateral surface $\nu_t = 0$, and we finally obtain

$$\int\limits_0^T \iint\limits_D \frac{\partial\theta}{\partial t}(u_x^2 + u_y^2)\,dxdydt = \int\limits_0^T \oint\limits_\delta u_t(-u_y\nu_x + u_x\nu_y)\,dldt.$$

We parameterize the curve δ by the arc length, i.e., $x = \delta^1(t), y = \delta^2(t)$. Then $\nu_x = \dot\delta^2, \nu_y = -\dot\delta^1$, and the equality takes the form

$$\int\limits_0^T \iint\limits_D \frac{\partial\theta}{\partial t}(u_x^2 + u_y^2)\,dxdydt =$$

$$= -\int\limits_0^T\int\limits_0^T \left[\frac{\partial u(x,y,t_2)}{\partial t_2}\left(u_x(x,y,t_2)\dot\delta^1(t_1) + u_y(x,y,t_2)\dot\delta^2(t_1)\right)\right]_{(x,y)=(\delta^1(t_1),\delta^2(t_1))} dt_1 dt_2 =$$

$$= -\int\limits_0^T\int\limits_0^T \frac{\partial u(\delta^1(t_1),\delta^2(t_1),t_2)}{\partial t_2}\frac{\partial u(\delta^1(t_1),\delta^2(t_1),t_2)}{\partial t_1}\,dt_1 dt_2.$$

We now use the boundary condition (1.1.5) to obtain

$$\int\limits_0^T \iint\limits_D \frac{\partial\theta}{\partial t}(u_x^2 + u_y^2)\,dxdydt = -\int\limits_0^T\int\limits_0^T \frac{\partial g(t_1,t_2)}{\partial t_2}\frac{\partial g(t_1,t_2)}{\partial t_1}\,dt_1 dt_2. \tag{1.1.7}$$

The latter formula implies uniqueness of a solution to our problem. Indeed, observe that $\partial\theta/\partial t > 0$. If $g \equiv 0$, then the formula implies that $u_x \equiv u_y \equiv 0$. With the help of equation (1.1.3), the latter relations imply that $f \equiv 0$.

We now obtain the stability estimate. To this end we square the both parts of equation (1.1.3):

$$(u_x\cos\theta + u_y\sin\theta)^2 = f^2(x,y).$$

Adding $(-u_x\sin\theta + u_y\cos\theta)^2$ to the both parts of the latter equality, we obtain

$$u_x^2 + u_y^2 = f^2 + (-u_x\sin\theta + u_y\cos\theta)^2.$$

This implies that

$$\frac{\partial\theta}{\partial t}f^2 \le \frac{\partial\theta}{\partial t}(u_x^2 + u_y^2).$$

Integrating the latter inequality, we obtain

$$\|f\|_{L_2(D)}^2 \le \frac{1}{2\pi}\int\limits_0^T \iint\limits_D \frac{\partial\theta}{\partial t}(u_x^2 + u_y^2)\,dxdydt.$$

Together with (1.1.7), the latter inequality gives

$$\|f\|_{L_2(D)}^2 \le -\frac{1}{2\pi}\int\limits_0^T\int\limits_0^T \frac{\partial g(t_1,t_2)}{\partial t_1}\frac{\partial g(t_1,t_2)}{\partial t_2}\,dt_1 dt_2. \tag{1.1.8}$$

Using the Cauchy — Bunjakovskiĭ inequality, we derive

$$\left[\int_0^T \int_0^T \frac{\partial g(t_1, t_2)}{\partial t_1} \frac{\partial g(t_1, t_2)}{\partial t_2} dt_1 dt_2\right]^2 \leq \int_0^T \int_0^T \left(\frac{\partial g(t_1, t_2)}{\partial t_1}\right)^2 dt_1 dt_2 \cdot \int_0^T \int_0^T \left(\frac{\partial g(t_1, t_2)}{\partial t_2}\right)^2 dt_1 dt_2.$$

Using the symmetry $g(t_1, t_2) = g(t_2, t_1)$, we obtain

$$\left|\int_0^T \int_0^T \frac{\partial g(t_1, t_2)}{\partial t_1} \frac{\partial g(t_1, t_2)}{\partial t_2} dt_1\, dt_2\right| \leq \int_0^T \int_0^T \left(\frac{\partial g(t_1, t_2)}{\partial t_1}\right)^2 dt_1 dt_2.$$

With the help of the latter inequality, (1.1.8) implies the estimate that is the claim of the theorem.

1.1.3 Local properties of the function u

Here we will prove the above-formulated properties (1–3) of the function u. To this end we rewrite the definition

$$u(x, y, t) = \int\limits_{\tilde{\gamma}(x,y,t)} f(x, y)\, ds$$

of the function u in the form

$$u(x, y, t) = \int\limits_0^{S(x,y,t)} f(\gamma^1(x, y, \theta(x, y, t) + \pi, \sigma), \gamma^2(x, y, \theta(x, y, t) + \pi, \sigma))\, d\sigma$$

where $S(x, y, t)$ is the length of $\tilde{\gamma}(x, y, t)$. Differentiating the latter equality, we get the formula

$$u_t = S_t f(\delta^1(t), \delta^2(t)) + \int\limits_0^{S(x,y,t)} (f_x \gamma_\theta^1 \theta_t + f_y \gamma_\theta^2 \theta_t) d\sigma$$

and the similar formulas for u_x, u_y which include $S_x, S_y, \gamma_x^i, \gamma_y^i, \theta_x, \theta_y$. Differentiating these formulas one more again, we have got expressions for the second derivatives $u_{xx}, u_{xy}, \ldots$ in terms of second derivatives of the functions $S(x, y, t), \theta(x, y, t)$ and $\delta^i(x, y, t)$. Therefore the above-formulated properties of the function u follow from the next claim.

Lemma 1.1.3 *The functions $S(x, y, t)$ and $\theta(x, y, t)$ belong to $C^2(\Omega \setminus \Omega_0)$. The derivatives S_x, S_y, S_t are bounded in $\Omega \setminus \Omega_0$, and the derivatives $\theta_x, \theta_y, \theta_t$ satisfy the estimates*

$$|\theta_x(x, y, t)| \leq \frac{C}{S(x, y, t)}, \quad |\theta_y(x, y, t)| \leq \frac{C}{S(x, y, t)}, \quad |\theta_t(x, y, t)| \leq \frac{C}{S(x, y, t)}.$$

Proof. The functions $S(x, y, t)$ and $\theta(x, y, t)$ are defined by the following system of equations

$$\gamma^1(x, y, \theta(x, y, t) + \pi, S(x, y, t)) = \delta^1(t), \quad \gamma^2(x, y, \theta(x, y, t) + \pi, S(x, y, t)) = \delta^2(t).$$

Differentiating these equalities with respect to t, we obtain

$$\gamma_\theta^1 \theta_t + \gamma_s^1 S_t = \dot{\delta}^1, \quad \gamma_\theta^2 \theta_t + \gamma_s^2 S_t = \dot{\delta}^2.$$

We consider the latter formulas as a linear system in θ_t and S_t. Solving the system by the Kramer rule, we obtain

$$\theta_t(x, y, t) = \frac{\Delta_1(x, y, \theta(x, y, t) + \pi, S(x, y, t))}{\Delta(x, y, \theta(x, y, t) + \pi, S(x, y, t))}, \tag{1.1.9}$$

$$S_t(x, y, t) = \frac{\Delta_2(x, y, \theta(x, y, t) + \pi, S(x, y, t))}{\Delta(x, y, \theta(x, y, t) + \pi, S(x, y, t))}. \tag{1.1.10}$$

where

$$\Delta(x, y, \theta, s) = \begin{vmatrix} \gamma_\theta^1 & \gamma_s^1 \\ \gamma_\theta^2 & \gamma_s^2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \dot{\delta}^1 & \gamma_s^1 \\ \dot{\delta}^2 & \gamma_s^2 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \gamma_\theta^1 & \dot{\delta}^1 \\ \gamma_\theta^2 & \dot{\delta}^2 \end{vmatrix}.$$

By the regularity condition, $\Delta(x, y, \theta, s) \geq Cs$. Therefore (1.1.9) implies the inequality $|\theta_t| \leq C/S$. Observe that the derivatives γ_θ^i admit the estimate $|\gamma_\theta^i| \leq Cs$ since $\gamma_\theta^i|_{s=0} = 0$. Therefore (1.1.10) implies boundedness of the derivative S_t. We thus have proved the claim on the derivative S_t. The claim on the other derivatives is proved in a similar way.

1.2   The nonlinear problem

Let D be a two-dimensional domain with the boundary $\delta = \partial D$ as in Section 1.1.1. Let $n(x,y) > 0$ be a function defined in D. We consider the Riemannian metric

$$d\tau^2 = n^2(x,y)(dx^2 + dy^2) \tag{1.2.1}$$

and the corresponding functional

$$J(\gamma) = \int_\gamma d\tau = \int_\gamma n(x,y)\sqrt{dx^2 + dy^2}. \tag{1.2.2}$$

The function n is assumed to be such that the family of extremals of functional (1.2.2) is regular in the sense of Section 1.1.1. In particular, for every two boundary points, the *boundary distance function*

$$\Gamma(t_1, t_2) = \int_{\gamma(t_1, t_2)} n\, ds \tag{1.2.3}$$

is defined; here $\gamma(t_1, t_2)$ is the unique extremal of functional (1.2.2) joining the points $\delta(t_1)$ and $\delta(t_2)$. We consider the problem of recovering the function $n(x,y)$ from the known boundary distance function $\Gamma(t_1, t_2)$.

Theorem 1.2.1 *Let a function $n \in C^4(\bar{D})$ be such that the corresponding family of extremals is regular. Then $n(x,y)$ can be uniquely recovered from $\Gamma(t_1, t_2)$, and the stability estimate*

$$\|n_1 - n_2\|_{L_2(D)} \le \frac{1}{\sqrt{2\pi}} \left\| \frac{\partial(\Gamma_1 - \Gamma_2)}{\partial t_1} \right\|_{L_2([0,T] \times [0,T])}$$

holds, where Γ_1 and Γ_2 are the boundary distance functions corresponding the functions n_1 and n_2.

Proof. We proceed in the same way as in Section 1.1.2. For $0 \le t \le T$ and $(x,y) \in D$, by $\gamma(x,y,t)$ we denote the extremal passing through the points $\delta(t) = (\delta^1(t), \delta^2(t))$ and (x,y), and by $\tilde{\gamma}(x,y,t)$ we denote the segment of the extremal between these points. Let $\theta(x,y,t)$ be the angle from the horizontal direction to the tangent vector of this extremal at the point (x,y). We introduce the function

$$\tau(x,y,t) = \int_{\tilde{\gamma}(x,y,t)} n(x,y)\sqrt{dx^2 + dy^2}. \tag{1.2.4}$$

For a fixed t, the curves $\tilde{\gamma}(x,y,t)$ form a family of extremals for functional (1.2.2) in the domain D. The Hamilton — Jacobi equation for the extremal family coincides with the eikonal equation

$$\tau_x^2 + \tau_y^2 = n^2(x,y). \tag{1.2.5}$$

By repeating the arguments used above for deriving (1.1.3), we obtain the equation

$$\tau_x \cos\theta + \tau_y \sin\theta = n(x,y). \tag{1.2.6}$$

Equations (1.2.5) and (1.2.6) imply that

$$\tau_x = n\cos\theta, \quad \tau_y = n\sin\theta. \tag{1.2.7}$$

Differentiating (1.2.5) with respect to t, we obtain

$$\frac{\partial}{\partial t}(\tau_x^2 + \tau_y^2) = 0. \tag{1.2.8}$$

The definition (1.2.4) implies the boundary condition

$$\tau(\delta^1(t_2), \delta^2(t_2), t_1) = \Gamma(t_1, t_2). \tag{1.2.9}$$

We now consider (1.2.8)–(1.2.9) as a boundary value problem for the function τ. Note that (1.2.8) is a nonlinear equation. Nevertheless, uniqueness of the solution to the problem can be proved by the same method is in Section 1.1.2.

Let $n_1(x, y)$ and $n_2(x, y)$ be two functions satisfying hypotheses of Theorem 1.2.1;
$\tau_1(x, y, t)$ and $\tau_2(x, y, t)$ be the corresponding eikonals;
$\Gamma_1(t_1, t_2)$ and $\Gamma_2(t_1, t_2)$ be the corresponding boundary distance functions.
We put $u(x, y, t) = \tau_1 - \tau_2$.

Substituting τ_1 and then τ_2 into (1.2.8) and taking the difference of the so-obtained equalities, we get the equation

$$\frac{\partial}{\partial t} \left[\left(\frac{\partial \tau_1}{\partial x} \right)^2 + \left(\frac{\partial \tau_1}{\partial y} \right)^2 - \left(\frac{\partial \tau_2}{\partial x} \right)^2 - \left(\frac{\partial \tau_2}{\partial y} \right)^2 \right] = 0.$$

This equation can be rewritten in the form

$$\frac{\partial}{\partial t} \left[\left(\frac{\partial \tau_1}{\partial x} - \frac{\partial \tau_2}{\partial x} \right) \left(\frac{\partial \tau_1}{\partial x} + \frac{\partial \tau_2}{\partial x} \right) + \left(\frac{\partial \tau_1}{\partial y} - \frac{\partial \tau_2}{\partial y} \right) \left(\frac{\partial \tau_1}{\partial y} + \frac{\partial \tau_2}{\partial y} \right) \right] = 0$$

or, in terms of the function u, in the form

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \frac{\partial \tau_1}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \tau_1}{\partial y} \right) + \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \frac{\partial \tau_2}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \tau_2}{\partial y} \right) = 0.$$

Using (1.2.7) and independence of n_i of t, we transform the latter equation to the form

$$\frac{\partial}{\partial t} \left(u_x \cos \theta_1 + u_y \sin \theta_1 \right) + \frac{n_2}{n_1} \frac{\partial}{\partial t} \left(u_x \cos \theta_2 + u_y \sin \theta_2 \right) = 0.$$

Multiplying the latter equality by $2(-u_x \sin \theta_1 + u_y \cos \theta_1)$, we obtain

$$2(-u_x \sin \theta_1 + u_y \cos \theta_1) \frac{\partial}{\partial t} \left(u_x \cos \theta_1 + u_y \sin \theta_1 \right) +$$

$$+2 \frac{n_2}{n_1} (-u_x \sin \theta_1 + u_y \cos \theta_1) \frac{\partial}{\partial t} \left(u_x \cos \theta_2 + u_y \sin \theta_2 \right) = 0. \qquad (1.2.10)$$

We transform the first summand of (1.2.10) by Lemma 1.1.2

$$2(-u_x \sin \theta_1 + u_y \cos \theta_1) \frac{\partial}{\partial t} \left(u_x \cos \theta_1 + u_y \sin \theta_1 \right) =$$

$$= \frac{\partial \theta_1}{\partial t} (u_x^2 + u_y^2) + \frac{\partial}{\partial x} (u_y u_t) - \frac{\partial}{\partial y} (u_x u_t) + \frac{\partial}{\partial t} \left[(-u_x \sin \theta_1 + u_y \cos \theta_1)(u_x \cos \theta_1 + u_y \sin \theta_1) \right]. \quad (1.2.11)$$

We will show that the second term on the left-hand side of (1.2.10) can be transformed to a divergent form. Using (1.2.7) again, we obtain

$$2 \frac{n_2}{n_1} (-u_x \sin \theta_1 + u_y \cos \theta_1) \frac{\partial}{\partial t} (u_x \cos \theta_2 + u_y \sin \theta_2) =$$

$$= 2 \frac{n_2}{n_1} \left[-(n_1 \cos \theta_1 - n_2 \cos \theta_2) \sin \theta_1 + (n_1 \sin \theta_1 - n_2 \sin \theta_2) \cos \theta_1 \right] \times$$

$$\times \frac{\partial}{\partial t} \left[(n_1 \cos \theta_1 - n_2 \cos \theta_2) \cos \theta_2 + (n_1 \sin \theta_1 - n_2 \sin \theta_2) \sin \theta_2 \right] =$$

$$= 2 \frac{n_2}{n_1} n_2 \sin(\theta_1 - \theta_2) \frac{\partial}{\partial t} \left(n_1 \cos(\theta_1 - \theta_2) - n_2 \right) = 2 n_2^2 \sin(\theta_1 - \theta_2) \frac{\partial}{\partial t} \cos(\theta_1 - \theta_2) =$$

$$= -2 n_2^2 \sin^2(\theta_1 - \theta_2) \frac{\partial}{\partial t} (\theta_1 - \theta_2) = -n_2^2 (1 - \cos 2(\theta_1 - \theta_2)) \frac{\partial}{\partial t} (\theta_1 - \theta_2).$$

We have thus obtained

$$2 \frac{n_2}{n_1} (-u_x \sin \theta_1 + u_y \cos \theta_1) \frac{\partial}{\partial t} (u_x \cos \theta_2 + u_y \sin \theta_2) = \frac{1}{2} \frac{\partial}{\partial t} \left(n_2^2 \sin 2(\theta_1 - \theta_2) \right) - \frac{\partial}{\partial t} \left(n_2^2 (\theta_1 - \theta_2) \right). \quad (1.2.12)$$

Inserting (1.2.11) and (1.2.12) into (1.2.10), we obtain

$$\frac{\partial \theta_1}{\partial t}(u_x^2 + u_y^2) = -\frac{\partial}{\partial x}(u_y u_t) + \frac{\partial}{\partial y}(u_x u_t)-$$

$$-\frac{\partial}{\partial t}\left[(-u_x \sin\theta_1 + u_y \cos\theta_1)(u_x \cos\theta_1 + u_y \sin\theta_1) - \frac{1}{2}n_2^2 \sin 2(\theta_1 - \theta_2) + n_2^2(\theta_1 - \theta_2)\right].$$

Now we integrate the latter equality over $D \times [0, T]$ and transform the right-hand side by the Gauss — Ostrogradskiĭ formula. The same difficulty as in Section 1.1.2 arises because of singularities of the functions $u(x, y, t)$ and $\theta_i(x, y, t)$ at $(x, y, t) = (\delta^1(t), \delta^2(t), t)$. This difficulty is overcame by the same arguments as in Section 1.1.2; we omit the details. As before, the integrals over upper and low bottoms cancel because of periodicity of integrands. We thus obtain

$$\int\limits_0^T \iint\limits_D \frac{\partial \theta_1}{\partial t}(u_x^2 + u_y^2)\, dxdydt = \int\limits_0^T \oint\limits_\delta u_t(-u_y \nu_x + u_x \nu_y)\, dldt.$$

Transforming the right-hand side integral in the same way as in Section 1.1.2, the equality obtains the form

$$\int\limits_0^T \iint\limits_D \frac{\partial \theta_1}{\partial t}(u_x^2 + u_y^2)\, dxdydt = \int\limits_0^T \int\limits_0^T \frac{\partial \Gamma(t_1, t_2)}{\partial t_1}\frac{\partial \Gamma(t_1, t_2)}{\partial t_2}\, dt_1 dt_2, \qquad (1.2.13)$$

where $\Gamma = \Gamma_1 - \Gamma_2$. Note that (1.2.13) implies uniqueness of a solution to the boundary value problem (1.2.8)–(1.2.9).

We will now obtain the stability estimate. From (1.2.5) and (1.2.7), we derive

$$u_x^2 + u_y^2 = \left(\frac{\partial \tau_1}{\partial x} - \frac{\partial \tau_2}{\partial x}\right)^2 + \left(\frac{\partial \tau_1}{\partial y} - \frac{\partial \tau_2}{\partial y}\right)^2 =$$

$$= \left(\frac{\partial \tau_1}{\partial x}\right)^2 + \left(\frac{\partial \tau_1}{\partial y}\right)^2 + \left(\frac{\partial \tau_2}{\partial x}\right)^2 + \left(\frac{\partial \tau_2}{\partial y}\right)^2 - 2\left(\frac{\partial \tau_1}{\partial x}\frac{\partial \tau_2}{\partial x} + \frac{\partial \tau_1}{\partial y}\frac{\partial \tau_2}{\partial y}\right) =$$

$$= n_1^2 + n_2^2 - 2n_1 n_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) =$$

$$= n_1^2 + n_2^2 - 2n_1 n_2 \cos(\theta_1 - \theta_2) = n_1^2 + n_2^2 - 2n_1 n_2\left(1 - 2\sin^2\frac{\theta_1 - \theta_2}{2}\right) = (n_1 - n_2)^2 + 4n_1 n_2 \sin^2\frac{\theta_1 - \theta_2}{2}.$$

Hence

$$(n_1 - n_2)^2 \le u_x^2 + u_y^2.$$

Multiplying the latter inequality by $\partial \theta_1/\partial t > 0$ and integrating it, we obtain

$$2\pi\|n_1 - n_2\|_{L_2(D)}^2 \le \int\limits_0^T \iint\limits_D \frac{\partial \theta_1}{\partial t}(u_x^2 + u_y^2)dxdydt. \qquad (1.2.14)$$

Relations (1.2.13) and (1.2.14) imply the stability estimate from the claim of the theorem in the same way as in Section 1.1.2.

1.3   Some remarks

The inverse kinematic problem of seismics has been investigated for a long time because its practical importance for geophysics. The first mathematical result on the problem was obtained by G. Herglotz, E. Wiechert and K. Zoeppritz in 1905 [38, 90]. In order to investigate a spherically symmetric model of Earth, they considered the following problem: one has to determine a positive function $n(r)$ $(0 \le r \le R)$ from the boundary distance function of the metric

$$d\tau^2 = n^2(r)|dx|^2 \qquad (1.3.1)$$

in the ball $\{x \in \mathbf{R}^3 \mid r^2 = |x|^2 = x_1^2 + x_2^2 + x_3^2 \le R^2\}$. They solved the problem under the assumption $(rn(r))' > 0$. The solution is based on the following fact that has been known before in differential geometry: the equation of geodesics on a surface of revolution admits a first integral (the Clairaut integral). Due to this fact, the boundary distance function and $n(r)$ are related by some integral equation

of Abelian type which is very good for numerical solving. This work has plaid an essential role in geophysics; the most of our knowledge on inner structure of Earth is due to this method.

Of course, Earth is spherically symmetric only in the zero approximation. Deviations from spherical symmetry (which are called horizontal nonhomogeneities in seismics) are very significant. Assuming the nonhomogeneities to be small, we can linearize the inverse kinematic problem near a known spherically symmetric metric. In such the way we arrive at the following integral geometry problem: one has to determine a function $f(x_1, x_2, x_3)$ in the ball $\{x \in \mathbf{R}^3 \mid |x| \leq R\}$ from its known integrals over all geodesics, of a given metric (1.3.1), joining boundary points. This problem was solved by V. G. Romanov [68]. His method is now very popular in practical seismics. We refer the reader to the excellent book by V. G. Romanov [70] which contains the mathematical introduction to the inverse kinematic problem of seismics as well as an extensive bibliography.

In a more complicated case, when the medium is not spherically symmetric, we have to recover a function $n(x_1, x_2, x_2)$ of three variables from the boundary distance function of the metric

$$d\tau^2 = n^2 |dx|^2 \tag{1.3.2}$$

In geophysics metrics of such type are called isotropic. Linearization of this problem in the class of isotropic metrics leads to linear integral geometry problems like the problem considered in Section 1.1. As we have mentioned, the first general results on these problems, linear and nonlinear, were obtained by R. G. Mukhometov in the two-dimensional case.

Since the papers [55, 56] by R. G. Mukhometov were published, many mathematicians looked for a multidimensional generalization. Finally R. G. Mukhometov himself [57, 58] as well as some other authors [10, 14, 69] have found the proof almost simultaneously. In contrast to the two-dimensional case, the method has obtained a rather complicated form in the multidimensional case, so every of the mentioned articles is not easy for reading. There is one more, somewhat mysterious, difference between the $2D$- and multidimensional cases: as we have seen, the linear problem can be considered for an arbitrary regular family of curves in the $2D$-case; while all known results on the multidimensional problem are obtained only for the family of geodesics of a Riemannian (or Finsler) metric.

In these lectures we develop some alternative approach to integral geometry which was suggested by L. N. Pestov and the author in [66]. I hope this approach is easier for understanding because of its geometrical background. Besides this, our method has the following two merits. First, it can be applied not only to isotropic metrics but to Riemannian metrics of general type. Second, the method fits not only integral geometry of scalar functions but also integral geometry of symmetric tensor fields of arbitrary degree. As we will see, an integral geometry problem for a tensor field of second degree arises in the process of linearization of the inverse kinematic problem in the class of arbitrary Riemannian metrics. On the other hand, integral geometry problems for vector and tensor fields arise in tomography of anisotropic media. The most known of such tomographic fields is Doppler tomography where one has to recover the velocity distribution of a fluid or gas from results of ultrasound measurements [41]. Photoelasticity [1, 74] gives us another example of tomographic problems for tensor fields of second degree.

Lecture 2
Some questions of tensor analysis

Here we present some notions and facts of tensor analysis that are used in the next lectures for investigating integral geometry of tensor fields on Riemannian manifolds.

The first three sections contain a survey of tensor algebra and theory of connections on manifolds including the definition of the Levi-Civita connection. This survey is not a systematic introduction to the subject; here the author's only purpose is to represent main notions and formulas in the form convenient for usage in the lectures. Our presentation of connection theory is nearest to that of the book [45]; although there are many other excellent textbooks on the subject [33, 51, 25].

In Section 2.4 the operators of inner differentiation and divergence are introduced on the space of symmetric tensor fields of a Riemannian manifold; their duality is established. Then we prove the theorem on decomposition of a tensor field, on a compact manifold, into the sum of solenoidal and potential fields.

Sections 2.5–2.7 are devoted to presenting main tools that are applied in the next lectures to studying integral geometry on Riemannian manifolds. The so-called semibasic tensor fields are defined on the space of the tangent bundle; two differential operators, vertical and horizontal covariant derivatives, are introduced on semibasic tensor fields. For these operators, formulas of Gauss-Ostrogradskiĭ type are established. The term "semibasic tensor field" is adopted from the book [31] in which a corresponding notion is considered for exterior differential forms.

The modern mathematical style presumes that invariant (independent of the choice of coordinates) notions are introduced by invariant definitions. In these lectures we consciously choose the opposite approach. The notions under consideration will always be introduced with the help of local coordinates. We will pay particular attention to the rule of transformation of the quantities under definition with respect to a change of coordinates. Such the style was widely used in geometry till 40-th; perhaps [82] is the most known book written in the style. Now this style is used only in physical literature. Our choice is caused by the following three reasons. First, these lectures are addressed to a wide mathematical audience, not only to geometers, and I would like to avoid abstract geometrical constructions, like vector or principal bundles, that are needed in an invariant presentation. Second, treating integral geometry in the next lectures, we will mostly use coordinate representation of tensor fields; and I am not sure that all our arguments can be presented in an invariant form. And finally, all our main results are obtained for topologically trivial manifolds that are diffeomorphic to an n-dimensional ball. Every bundle over such a manifold is trivial, and in fact we do not need to use any bundle.

The reader is assumed to be familiar with the definition of a manifold (possibly with boundary) and a local coordinate system (or chart) on a manifold. However, if he/she does not know the definition of a manifold, it is possible to think about a domain in Euclidean space (with smooth boundary) instead of a manifold.

For simplicity we will restrict ourselves to considering real tensor analysis. The only exception is Section 2.3 where we will need complex tensors. So, all functions, vector and tensor fields are assumed to be real if otherwise is not stated.

2.1 Tensor fields

Given a manifold M and an open set $U \subset M$, by $C^\infty(U)$ we denote the algebra of smooth functions on U. The term "smooth" is used as a synonym of "infinitely differentiable."

There are several equivalent definitions of a tangent vector of a manifold. One of them is as follows: a *tangent vector* to a manifold M at a point $x \in M$ is a linear mapping $v : C^\infty(M) \to \mathbf{R}$ satisfying the condition $v(fg) = f(x) \cdot vg + g(x) \cdot vf$. The set of all vectors tangent to M at a fixed point x constitutes an n-dimensional vector space ($n = \dim M$) which is denoted by $T_x M$. If $(x^1, \ldots, x^n)$ is a local coordinate

system on M with the domain $U \subset M$ then, for $x \in U$, the *coordinate vectors* $\partial_i = \frac{\partial}{\partial x^i}$ $(1 \leq i \leq n)$, defined by the equality $\partial_i f = \frac{\partial f}{\partial x^i}(x)$, constitute the basis of $T_x M$, i.e., every vector $v \in T_x M$ can be uniquely represented in the form

$$v = v^i \frac{\partial}{\partial x^i} \tag{2.1.1}$$

We use the following rule: on repeating sub- and super-indices in a monomial the summation from 1 to n is assumed.

A *vector field* v on a manifold M is a function that associates a vector $v(x) \in T_x M$ to every point $x \in M$ which is smooth in the following sense: if we represent, in the domain of a local coordinate system, v in form (2.1.1), then the coefficients $v^i = v^i(x)$ belong to $C^\infty(U)$. These coefficients are called *coordinates* or *components* of the vector field v with respect to the given coordinate system.

If $(x'^1, \ldots, x'^n)$ is another local coordinate system on M with the domain $U' \subset M$, then the same vector field v has the components $v'^i \in C^\infty(U')$ with respect to this coordinate system. In $U \cap U'$ the two families of components are related by the equalities

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j. \tag{2.1.2}$$

This formula can be used as a base of the following definition of a vector field which is equivalent to the previous one: a vector field v on a manifold M is a rule associating a family of functions $v^i \in C^\infty(U)$ $(1 \leq i \leq n)$ to every local coordinate system with the domain U which are transformed by formula (2.1.2) under a change of coordinates.

Given an open set $U \subset M$, by $C^\infty(\tau_M; U)$ we denote the set of all vector fields on U. The notation $C^\infty(\tau_M; M)$ will be abbreviated to $C^\infty(\tau_M)$. It is the standard notation that is explained as follows: τ_M is the tangent bundle of the manifold M (we did not introduce its definition), and $C^\infty(\tau_M)$ is the space of smooth sections of the bundle. $C^\infty(\tau_M)$ is the $C^\infty(M)$-module, i.e., vector fields can be summed and multiplied by smooth functions. Given a local coordinate system $(x^1, \ldots, x^n)$ with the domain U, the *coordinate vector fields* $\partial_i = \frac{\partial}{\partial x^i}$ $(1 \leq i \leq n)$ constitute the basis of the $C^\infty(U)$-module $C^\infty(\tau_M; U)$.

As is seen from (2.1.1), a vector field $v \in C^\infty(\tau_M)$ can be considered as a *derivative* of the ring $C^\infty(M)$, i.e., as an **R**-linear mapping $v : C^\infty(M) \to C^\infty(M)$ such that $v(f \cdot g) = vf \cdot g + f \cdot vg$. The number $vf(x)$ is called the *derivative of the function f at the point x in the direction $v(x)$*. In fact, this is the new definition of a vector field equivalent to the previous two ones. For two such derivatives v and w, their commutator $[v, w] = vw - wv$ is also a derivative, i.e., a vector field. It is called the *Lie commutator* of the fields v and w.

Given a function $f \in C^\infty(M)$ and local coordinate system $(x^1, \ldots, x^n)$ with the domain $U \subset M$, let us consider the family of partial derivatives $v_i = \frac{\partial f}{\partial x^i} \in C^\infty(U)$ $(1 \leq i \leq n)$. Under a change of coordinates the family is transformed by the rule:

$$v'_i = \frac{\partial x^j}{\partial x'_i} v_j \tag{2.1.3}$$

We emphasize that formulas (2.1.2) and (2.1.3) are different. We use the latter formula as a base of the following definition: A *covector field* v on a manifold M is a rule associating a family of functions $v_i \in C^\infty(U)$ $(1 \leq i \leq n)$ to every local coordinate system with the domain U which are transformed by formula (2.1.3) under a change of coordinates. The $C^\infty(M)$-module of all covector fields is denoted by $C^\infty(\tau'_M)$ because it is the space of sections of the cotangent bundle τ'_M that is dual to τ_M. Covector fields are also called *one-forms*.

For a function $f \in C^\infty(M)$, the *differential* $df \in C^\infty(\tau'_M)$ of f is correctly defined by putting $(df)_i = \frac{\partial f}{\partial x^i} \in C^\infty(U)$ for the domain U of a coordinate system $(x^1, \ldots, x^n)$. In particular, we can consider the differentials $dx^i \in C^\infty(\tau'_M; U)$ $(1 \leq i \leq n)$ of the coordinate functions. These fields are called the *coordinate covector fields*. They constitute the basis of the $C^\infty(U)$-module $C^\infty(\tau'_M; U)$, i.e., every covector field $v \in C^\infty(\tau'_M; U)$ can be uniquely represented in the form

$$v = v_i dx^i \tag{2.1.4}$$

By analogy with (2.1.2) and (2.1.3), we give the following definition. Given nonnegative integers r and s, a *tensor field* v of degree (r, s) on a manifold M is a rule associating a family of functions $v^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in C^\infty(U)$ (all indices vary from 1 to n) to every local coordinate system with the domain U which are transformed by the formula

$$v'^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x'^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial x'^{j_s}} v^{k_1 \ldots k_r}_{l_1 \ldots l_s} \tag{2.1.5}$$

under a change of coordinates. Given an open set $U \subset M$, by $C^\infty(\tau_s^r M; U)$ we denote the set of all tensor fields of degree (r, s) on U. The notation $C^\infty(\tau_s^r M; M)$ will be abbreviated to $C^\infty(\tau_s^r M)$. A tensor field $v \in C^\infty(\tau_s^r M)$ is said to be r times contravariant and s times covariant.

Tensor fields of degrees $(0,0)$, $(1,0)$ and $(0,1)$ are just smooth functions, vector and covector fields respectively.

We will now list some algebraic operations which are defined on tensor fields.

$C^\infty(\tau_s^r M; U)$ is a $C^\infty(U)$-module, i.e., tensor fields of the same degree can be summed and multiplied by smooth functions $f \in C^\infty(U)$.

Every permutation π of the set $\{1, \ldots, r\}$ (of the set $\{1, \ldots, s\}$) determines the automorphism ρ^π (automorphism ρ_π) of the module $C^\infty(\tau_s^r M)$ by the formulas

$$(\rho^\pi v)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = v^{i_{\pi(1)} \ldots i_{\pi(r)}}_{j_1 \ldots j_s}, \quad (\rho_\pi v)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = v^{i_1 \ldots i_r}_{j_{\pi(1)} \ldots j_{\pi(s)}}. \tag{2.1.6}$$

The automorphism ρ^π (ρ_π) is called the *operator of transposition* of upper (lower) indices.

For $1 \le k \le r$ and $1 \le l \le s$ the *contraction operator* $C_l^k : C^\infty(\tau_s^r M) \to C^\infty(\tau_{s-1}^{r-1} M)$ with respect to k-th upper and l-th lower indices is defined by the equality

$$(C_l^k v)^{i_1 \ldots i_{r-1}}_{j_1 \ldots j_{s-1}} = v^{i_1 \ldots i_{k-1} p i_k \ldots i_{r-1}}_{j_1 \ldots j_{l-1} p j_l \ldots j_{s-1}}. \tag{2.1.7}$$

Given $v \in C^\infty(\tau_s^r M)$ and $w \in C^\infty(\tau_{s'}^{r'} M)$, the *tensor product* $v \otimes w \in C^\infty(\tau_{s+s'}^{r+r'} M)$ is defined by the formula

$$(v \otimes w)^{i_1 \ldots i_{r+r'}}_{j_1 \ldots j_{s+s'}} = v^{i_1 \ldots i_r}_{j_1 \ldots j_s} w^{i_{r+1} \ldots i_{r+r'}}_{j_{s+1} \ldots j_{s+s'}}. \tag{2.1.8}$$

This product turns $C^\infty(\tau_*^* M) = \bigoplus_{r,s=0}^\infty C^\infty(\tau_s^r M)$ into a bigraded $C^\infty(M)$-algebra.

If $(x^1, \ldots, x^n)$ is a local coordinate system with the domain $U \subset M$, then any tensor field $v \in C^\infty(\tau_s^r M; U)$ can be uniquely represented as

$$v = v^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}. \tag{2.1.9}$$

where $v^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in C^\infty(U)$ are called the *coordinates* (or the *components*) of the field v in the given coordinate system. Assuming the choice of coordinates to be clear from the context, we will usually abbreviate equality (2.1.9) as follows:

$$v = (v^{i_1 \ldots i_r}_{j_1 \ldots j_s}). \tag{2.1.10}$$

Note that the tensor fields $\partial/\partial x^i$ and dx^j commute with respect to the tensor product, i.e., $\partial/\partial x^i \otimes dx^j = dx^j \otimes \partial/\partial x^i$, while dx^i and dx^j (and also $\partial/\partial x^i$ and $\partial/\partial x^j$) do not commute. Moreover, if U is diffeomorphic to $\mathbf{R}^n$, then the $C^\infty(U)$-algebra $C^\infty(\tau_*^* M; U)$ is obtained from the free $C^\infty(U)$-algebra with generators $\partial/\partial x^i$ and dx^i by the defining relations $\partial/\partial x^i \otimes dx^j = dx^j \otimes \partial/\partial x^i$.

Using the pairing $\langle v, w \rangle = v^{i_1 \ldots i_r}_{j_1 \ldots j_s} w^{j_1 \ldots j_s}_{i_1 \ldots i_r}$, we can consider $C^\infty(\tau_s^r M)$ and $C^\infty(\tau_r^s M)$ as the mutually dual $C^\infty(M)$-modules. This implies, in particular, that a covariant tensor field $v \in C^\infty(\tau_s^0 M)$ can be considered as a $C^\infty(M)$-multilinear mapping $v : C^\infty(\tau_M) \times \ldots \times C^\infty(\tau_M) \to C^\infty(M)$. Similarly, a field $v \in C^\infty(\tau_s^1 M)$ can be considered as a $C^\infty(M)$-multilinear mapping $v : C^\infty(\tau_M) \times \ldots \times C^\infty(\tau_M) \to C^\infty(\tau_M)$.

2.2 Covariant differentiation

A *connection* on a manifold M is a mapping $\nabla : C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$ sending a pair of vector fields u, v into the third vector field $\nabla_u v$ which is $\mathbf{R}$-linear in the second argument, and $C^\infty(M)$-linear in the first argument, while satisfying the relation:

$$\nabla_u(\varphi v) = \varphi \nabla_u v + (u\varphi) v \tag{2.2.1}$$

for $\varphi \in C^\infty(M)$.

By one of remarks in the previous section, $C^\infty(\tau_1^1 M)$ is canonically identified with the set of $C^\infty(M)$-linear mappings $C^\infty(\tau_M) \to C^\infty(\tau_M)$. Consequently, a given connection defines the $\mathbf{R}$-linear mapping (which is denoted by the same letter)

$$\nabla : C^\infty(\tau_M) \to C^\infty(\tau_1^1 M) \tag{2.2.2}$$

by the formula $(\nabla v)(u) = \nabla_u v$. Relation (2.2.1) is rewritten as:

$$\nabla(\varphi v) = \varphi \cdot \nabla v + v \otimes d\varphi. \tag{2.2.3}$$

The tensor field ∇v is called the *covariant derivative* of the vector field v (with respect to the given connection).

The covariant differentiation, having been defined on vector fields, can be transferred to tensor fields of arbitrary degree, as the next theorem shows.

Theorem 2.2.1 *Given a connection, there exist uniquely determined* **R**-*linear mappings*

$$\nabla : C^\infty(\tau_s^r M) \to C^\infty(\tau_{s+1}^r M), \tag{2.2.4}$$

for all integers r and s, such that
  (1) $\nabla \varphi = d\varphi$ *for* $\varphi \in C^\infty(M) = C^\infty(\tau_0^0 M)$;
  (2) *For $r = 1$ and $s = 0$, mapping (2.2.4) coincides with the above-defined mapping (2.2.2)*;
  (3) *For $1 \le k \le r$ and $1 \le l \le s$, operator (2.2.4) commutes with the contraction operator C_l^k*;
  (4) *the operator ∇ is a derivative of the algebra $C^\infty(\tau_*^* M)$ in the following sense: for $u \in C^\infty(\tau_s^r M)$ and $v \in C^\infty(\tau_{s'}^{r'} M)$,*

$$\nabla(u \otimes v) = \rho_{s+1}(\nabla u \otimes v) + u \otimes \nabla v, \tag{2.2.5}$$

where ρ_{s+1} is the transposition operator for lower indices corresponding to the permutation $\{1, \ldots, s, s + 2, \ldots, s + s' + 1, s + 1\}$.

We omit the proof of the theorem which can be accomplished in a rather elementary way based on the local representation (2.1.9).

For a connection ∇ the mappings $R : C^\infty(\tau_M) \times C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$ and $T : C^\infty(\tau_M) \times C^\infty(\tau_M) \to C^\infty(\tau_M)$ defined by the formulas

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

$$T(u, v) = \nabla_u v - \nabla_v u - [u, v],$$

are $C^\infty(M)$-linear in all arguments and, consequently, they are tensor fields of degrees (1,3) and (1,2) respectively. They are called the *curvature tensor* and the *torsion tensor* of the connection ∇. A connection with the vanishing torsion tensor is called *symmetric*.

Let us present the coordinate form of the covariant derivative. If $(x^1, \ldots, x^n)$ is a local coordinate system defined in a domain $U \subset M$, then the *Christoffel symbols* of the connection ∇ are defined by the equalities

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \tag{2.2.6}$$

where $\partial_i = \partial/\partial x^i$ are the coordinate vector fields. We emphasize that the functions $\Gamma_{ij}^k \in C^\infty(U)$ are not components of any tensor field; under a change of the coordinates, they are transformed by the formulas

$$\Gamma'^k_{ij} = \frac{\partial x'^k}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^i} \frac{\partial x^\gamma}{\partial x'^j} \Gamma_{\beta\gamma}^\alpha + \frac{\partial x'^k}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^i \partial x'^j}. \tag{2.2.7}$$

The torsion tensor and curvature tensor are expressed through the Christoffel symbols by the equalities

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k,$$

$$R^k{}_{lij} = \frac{\partial}{\partial x^i} \Gamma_{jl}^k - \frac{\partial}{\partial x^j} \Gamma_{il}^k + \Gamma_{ip}^k \Gamma_{jl}^p - \Gamma_{jp}^k \Gamma_{il}^p. \tag{2.2.8}$$

For a field

$$u = u_{j_1 \ldots j_s}^{i_1 \ldots i_r} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s},$$

the components of the field ∇u are denoted by $u_{j_1 \ldots j_s \, ; \, k}^{i_1 \ldots i_r}$ or by $\nabla_k u_{j_1 \ldots j_s}^{i_1 \ldots i_r}$, i.e.,

$$\nabla u = u_{j_1 \ldots j_s \, ; \, k}^{i_1 \ldots i_r} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^k.$$

We emphasize that the factor dx^k, corresponding to the number of the coordinate with respect to which "the differentiation is taken", is situated in the final position. Of course, this rule is not obligatory, but some choice must be done. Our choice stipulates the appearance the operator ρ_{s+1} in equality (2.2.5).

According to our choice, the notation $u^{i_1...i_r}_{j_1...j_s;\,k}$ is preferable to $\nabla_k u^{i_1...i_r}_{j_1...j_s}$, since it has the index k in the final position. Nevertheless, we will also use the second notation because it is convenient to interpret ∇_k as "the covariant partial derivative". The components of the field ∇u are expressed through the components of u by the formulas

$$\nabla_k u^{i_1...i_r}_{j_1...j_s} = u^{i_1...i_r}_{j_1...j_s;\,k} = \frac{\partial}{\partial x^k} u^{i_1...i_r}_{j_1...j_s} + \sum_{\alpha=1}^{r} \Gamma^{i_\alpha}_{kp} u^{i_1...i_{\alpha-1}pi_{\alpha+1}...i_r}_{j_1...j_s} - \sum_{\alpha=1}^{s} \Gamma^{p}_{kj_\alpha} u^{i_1...i_r}_{j_1...j_{\alpha-1}pj_{\alpha+1}...j_s}. \qquad (2.2.9)$$

The second-order covariant derivatives satisfy the commutation relations:

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) u^{i_1...i_r}_{j_1...j_s} = \sum_{\alpha=1}^{r} R^{i_\alpha}_{pkl} u^{i_1...i_{\alpha-1}pi_{\alpha+1}...i_r}_{j_1...j_s} - \sum_{\alpha=1}^{s} R^{p}_{j_\alpha kl} u^{i_1...i_r}_{j_1...j_{\alpha-1}pj_{\alpha+1}...j_s}. \qquad (2.2.10)$$

2.3 Riemannian manifolds

A *Riemannian metric* on a manifold M is a tensor field $g = (g_{ij}) \in C^\infty(\tau^0_2 M)$ such that the matrix $(g_{ij}(x))$ is symmetric and positive-definite for every point $x \in M$. A manifold M together with a fixed Riemannian metric is called *Riemannian manifold*. We denote a Riemannian manifold by (M, g) or simply by M if it is clear what metric is assumed. Given $\xi, \eta \in T_x M$, by $\langle \xi, \eta \rangle = g_{ij}(x)\xi^i \eta^j$ we mean the inner product.

A Riemannian metric defines the canonical isomorphisms of the $C^\infty(M)$-modules $C^\infty(\tau^r_s M) \cong C^\infty(\tau^{s+r}_0 M) \cong C^\infty(\tau^0_{s+r} M)$ which are considered as identifications. Due to these identifications, we will not distinguish co- and contravariant tensor fields on a Riemannian manifold and will speak about co- and contravariant coordinates of the same tensor field. In coordinate form this fact is expressed by the well-known rules of raising and lowering indices of a tensor field:

$$u_{i_1...i_m} = g_{i_1 j_1} \cdots g_{i_m j_m} u^{j_1...j_m}; \quad u^{i_1...i_m} = g^{i_1 j_1} \cdots g^{i_m j_m} u_{j_1...j_m},$$

where (g^{ij}) is the matrix inverse to (g_{ij}).

The inner product is extendible to the mapping

$$C^\infty(\tau^0_m M) \times C^\infty(\tau^0_m M) \to C^\infty(M), \quad (u, v) \mapsto \langle u, v \rangle$$

which is defined in coordinates by the equality

$$\langle u, v \rangle = u_{i_1...i_m} v^{i_1...i_m}. \qquad (2.3.1)$$

The latter allows us define the scalar product

$$(u, v)_{L_2(\tau^0_m M)} = \int_M \langle u, v \rangle(x)\, dV^n(x) \qquad (2.3.2)$$

on the space $C^\infty_0(\tau^0_m M)$ of compactly supported tensor fields. Here

$$dV^n(x) = [\det(g_{ij})]^{1/2} dx^1 \wedge \ldots \wedge dx^n \qquad (2.3.3)$$

is the Riemannian volume form on M.

A connection ∇ on a Riemannian manifold is called *compatible with the metric* if $v\langle \xi, \eta \rangle = \langle \nabla_v \xi, \eta \rangle + \langle \xi, \nabla_v \eta \rangle$ for every vector fields $v, \xi, \eta \in C^\infty(\tau_M)$. It is known that on a Riemannian manifold there is a unique symmetric connection compatible with the metric; it is called the *Levi-Civita connection*. Its Christoffel symbols are expressed through the components of the metric tensor by the formulas

$$\Gamma^k_{ij} = \frac{1}{2} g^{kp} \left(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p} \right). \qquad (2.3.4)$$

From now on we will use only this connection on a Riemannian manifold, unless we state otherwise.

A smooth mapping $\gamma : (a, b) \to M$ is called a *(parameterized) curve* in the manifold. In the domain of a local coordinate system a curve is given by equalities $x^i = \gamma^i(t)$ ($1 \le i \le n$). A *vector field along the curve* γ is a mapping associating to every $t \in (a, b)$ a vector $v \in T_{\gamma(t)} M$ which is smooth in an obvious sense. In local coordinates such a vector field is given by equalities $v = v^i(t)\frac{\partial}{\partial x^i}$. The space of all vector fields along γ is denoted by $C^\infty(\gamma^* \tau_M)$. In particular, the vector field $\dot{\gamma} \in C^\infty(\gamma^* \tau_M)$ along γ defined by

the equalities $\dot{\gamma}^i = d\gamma^i/dt$ is called the *speed vector field of the curve*. A connection ∇ on M induces the *operator of total differentiation* $D/dt : C^\infty(\gamma^*\tau_M) \to C^\infty(\gamma^*\tau_M)$ along γ which is defined in coordinate form by the equality $D/dt = \dot{\gamma}^i\nabla_i$. A vector field v along γ is called *parallel along* γ if $Dv/dt = 0$. If v is parallel along γ, we say that the vector $v(\gamma(t))$ is obtained from $v(\gamma(0))$ by *parallel transport along* γ.

A curve γ in a Riemannian manifold is called a *geodesic* if its speed field $\dot{\gamma}$ is parallel along γ. In coordinate form the equations of geodesics are

$$\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = 0. \tag{2.3.5}$$

Given a Riemannian manifold M without boundary, a geodesic $\gamma : (a, b) \to M$ ($-\infty \leq a < b \leq \infty$) is called *maximal* if it is not extendible to a geodesic $\gamma' : (a - \varepsilon_1, b + \varepsilon_2) \to M$ where $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$ and $\varepsilon_1 + \varepsilon_2 > 0$. It is known that there is a unique geodesic issuing from any point in any direction. More exactly, for every $x \in M$ and $\xi \in T_xM$, there exists a unique maximal geodesic $\gamma_{x,\xi} : (a, b) \to M$ ($-\infty \leq a < 0 < b \leq \infty$) such that the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \xi$ are satisfied. In geometry the notation $\exp_x(t\xi)$ is widely used instead of $\gamma_{x,\xi}(t)$, but the notation $\gamma_{x,\xi}(t)$ is more convenient for our purposes and it will be always used in the lectures.

Let $R = (R_{ijkl})$ be the curvature tensor of a Riemannian manifold M. For a point $x \in M$ and a two-dimensional subspace $\sigma \subset T_xM$, the number

$$K(x, \sigma) = R_{ijkl}\xi^i\xi^k\eta^j\eta^l / \left(|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2\right) \tag{2.3.6}$$

is independent of the choice of the basis ξ, η for σ. It is called the *sectional curvature* of the manifold M at the point x and in the two-dimensional direction σ. This notion is very popular in differential geometry [33].

A vector field $Y(t)$ along a geodesic $\gamma(t)$ is called a *Jacobi vector field* if it satisfies the Jacobi equation

$$\frac{D^2Y}{dt^2} + R(\dot{\gamma}, Y)\dot{\gamma} = 0.$$

2.4 Symmetric tensor fields

By $C^\infty(S^m\tau'_M)$ we denote the submodule of $C^\infty(\tau^0_mM)$ which consists of tensor fields invariant with respect to all transpositions of the indices. The notation is explained as follows: $S^m\tau'_M$ is the m-th symmetric power of the bundle τ'_M. We call elements of $C^\infty(S^*\tau'_M) = \bigoplus_{m=0}^\infty C^\infty(S^m\tau'_M)$ *symmetric tensor fields*. Let $\sigma : C^\infty(\tau^0_mM) \to C^\infty(S^m\tau'_M)$ be the *canonical projection* (*symmetrization*) defined by the equality $\sigma = \dfrac{1}{m!}\sum_{\pi\in\Pi_m}\rho_\pi$ where Π_m is the group of all permutations of degree m. The *symmetric product* $uv = \sigma(u \otimes v)$ turns $C^\infty(S^*\tau'_M)$ into the commutative graded $C^\infty(M)$-algebra.

From now on we assume in this section that M is a Riemannian manifold. Given $u \in C^\infty(S^m\tau'_M)$, by $i_u : C^\infty(S^l\tau'_M) \to C^\infty(S^{l+m}\tau'_M)$ we denote the operator of symmetric multiplication by u, and by j_u we denote the formally dual of i_u with respect to the inner product (2.3.2). In coordinate form these operators are expressed by formulas

$$(i_uv)_{i_1\ldots i_{l+m}} = \sigma(u_{i_1\ldots i_l}v_{i_{l+1}\ldots i_{l+m}}), \quad (j_uv)_{i_1\ldots i_{m-l}} = v_{i_1\ldots i_m}u^{i_{m-l+1}\ldots i_m},$$

The *operator of inner differentiation*

$$d : C^\infty(S^m\tau'_M) \to C^\infty(S^{m+1}\tau'_M) \tag{2.4.1}$$

is defined by the equality $d = \sigma\nabla$. The *divergence operator* $\delta : C^\infty(S^{m+1}\tau'_M) \to C^\infty(S^m\tau'_M)$ is defined in coordinate form by the formula $(\delta u)_{i_1\ldots i_m} = u_{i_1\ldots i_mj\,;\,k}g^{jk}$.

Theorem 2.4.1 *The operators d and $-\delta$ are formally dual to one other. Moreover, for a compact domain $D \subset M$ bounded by a piecewise smooth hypersurface ∂D and for every fields $u, v \in C^\infty(S^*\tau'_M)$, the following Green formula is valid:*

$$\int\limits_D [\langle du, v\rangle + \langle u, \delta v\rangle]\, dV^n = \int\limits_{\partial D} \langle i_\nu u, v\rangle\, dV^{n-1}, \tag{2.4.2}$$

where dV^n and dV^{n-1} are the Riemannian volumes on M and ∂D respectively, ν is the outer unit normal vector to ∂D.

Proof. It is known that, for a vector field $\xi \in C^\infty(\tau_M)$, the next Gauss-Ostrogradskiĭ formula is valid:

$$\int_D (\delta\xi)\, dV^n = \int_D \xi^i{}_{;\,i}\, dV^n = \int_{\partial D} \langle \xi, \nu \rangle\, dV^{n-1}. \tag{2.4.3}$$

Given $u \in C^\infty(S^m\tau'_M)$ and $v \in C^\infty(S^{m+1}\tau'_M)$, we write

$$\langle du, v\rangle + \langle u, \delta v\rangle = u_{i_1\dots i_m\,;\,i_{m+1}} v^{i_1\dots i_{m+1}} + u_{i_1\dots i_m} v^{i_1\dots i_{m+1}}{}_{;\,i_{m+1}} = \left(u_{i_1\dots i_m} v^{i_1\dots i_{m+1}} \right)_{;\,i_{m+1}}.$$

Introducing the vector field ξ by the equality $\xi^j = u_{i_1\dots i_m} v^{i_1\dots i_m j}$ and applying (2.4.3) to ξ, we arrive at (2.4.2).

Let (M, g) be a compact Riemannian manifold. For an integer $k \geq 0$, we define the real Hilbert space $H^k(S^m\tau'_M)$ as a completion of $C^\infty(S^m\tau'_M)$ with respect to the Sobolev norm $\|\cdot\|_k$ corresponding to the scalar product $(\cdot, \cdot)_k$ that is defined inductively in k by the formula

$$(u, v)_k = (\nabla u, \nabla v)_{k-1} + (u, v)_{L_2}.$$

In particular, $H^0(S^m\tau'_M) = L_2(S^m\tau'_M)$.

The next theorem generalizes the well-known fact about decomposition of a vector field ($m = 1$) into potential and solenoidal parts to symmetric tensor fields of arbitrary degree.

Theorem 2.4.2 *Let M be a compact Riemannian manifold with boundary; let $k \geq 1$ and $m \geq 0$ be integers. For every field $f \in H^k(S^m\tau'_M)$, there exist uniquely determined ${}^s\!f \in H^k(S^m\tau'_M)$ and $v \in H^{k+1}(S^{m-1}\tau'_M)$ such that*

$$f = {}^s\!f + dv, \quad \delta\, {}^s\!f = 0, \quad v|_{\partial M} = 0. \tag{2.4.4}$$

The estimates

$$\|{}^s\!f\|_k \leq C\|f\|_k, \quad \|v\|_{k+1} \leq C\|\delta f\|_{k-1} \tag{2.4.5}$$

are valid where a constant C is independent of f. In particular, ${}^s\!f$ and v are smooth if f is smooth.

We call the fields ${}^s\!f$ and dv the *solenoidal* and *potential* parts of the field f.

Proof. Here we will use a little bit of terminology from vector bundle theory.

Let $S^m\tau'_M|_{\partial M}$ be the restriction of the bundle $S^m\tau'_M$ to ∂M. We recall that, for $k \geq 1$, the trace operator $H^k(S^m\tau'_M) \to H^{k-1}(S^m\tau'_M|_{\partial M})$, $u \mapsto u|_{\partial M}$ is bounded.

Assuming existence of ${}^s\!f$ and v which satisfy (2.4.4) and applying the operator δ to the first of these equalities, we see that v is a solution to the boundary value problem $\delta dv = \delta f$, $v|_{\partial M} = 0$. Conversely, if we establish that, for any $u \in H^{k-1}(S^m\tau'_M)$, the boundary value problem

$$\delta dv = u, \quad v|_{\partial M} = 0 \tag{2.4.6}$$

has a unique solution $v \in H^{k+1}(S^m\tau'_M)$ satisfying the estimate

$$\|v\|_{k+1} \leq C\|u\|_{k-1}, \tag{2.4.7}$$

then we shall arrive at the claim of the theorem by putting $u = \delta f$ and ${}^s\!f = f - dv$.

We will show that problem (2.4.6) is elliptic with zero kernel and zero cokernel. After this, applying the theorem on normal solvability [89], we shall obtain existence and uniqueness of the solution to problem (2.4.6) as well as estimate (2.4.7).

To check ellipticity of problem (2.4.6) we have to show that the symbol $\sigma_2(\delta d)$ of the operator δd is elliptic and to verify the Lopatinskiĭ condition for the problem.

We use the definition and notation, for symbols of differential operators on vector bundles, that are given in [65]. It is straightforward from the definition that the symbols of operators d and δ are expressed by the formulas

$$\sigma_1 d(x, \xi) = i_\xi, \quad \sigma_1 \delta(x, \xi) = j_\xi \quad (\xi \in T'_x M),$$

where i_ξ and j_ξ are the operators defined in the beginning of the current section. Thus, $\sigma_2(\delta d)(\xi, u) = j_\xi i_\xi u$. Now we use the next

Lemma 2.4.3 *Let M be a Riemannian manifold, $x \in M$ and $0 \neq \xi \in T'_x M$. For an integer $m \geq 0$, the equality*

$$j_\xi i_\xi = \frac{1}{(m+1)}|\xi|^2 E + \frac{m}{(m+1)} i_\xi j_\xi \tag{2.4.8}$$

holds on the fiber $S^m T'_x M$ of the bundle $S^m\tau'_M$, where E is the identity operator.

The lemma will be proved at the end of the section, and now we continue the proof of the theorem. The operator $i_\xi j_\xi$ is nonnegative, as a product of two mutually dual operators. Consequently, formula (2.4.8) implies positiveness of $j_\xi i_\xi$ for $\xi \neq 0$. Thus ellipticity of the symbol $\sigma_2(\delta d)$ is proved.

It will be convenient for us to verify the Lopatinskiĭ condition in the form presented in [89] (condition III of this paper; we note simultaneously that condition II of regular ellipticity is satisfied since equation (2.4.6) has real coefficients). We choose local coordinates $(x^1, \ldots, x^{n-1}, x^n = t \geq 0)$ in a neighborhood of a point $x_0 \in \partial M$ in such a way that the boundary ∂M is determined by the equation $t = 0$ and $g_{ij}(x_0) = \delta_{ij}$. For brevity we denote $d_0(D) = \sigma_1 d(x_0, D)$ and $\delta_0(D) = \sigma_1 \delta(x_0, D)$ where $D = (D_j)$, $D_j = -i\partial/\partial x^j$ (Because of presence of the imaginary unit i, we should consider tensors and tensor fields with complex components in this section). Then

$$(d_0(D)v)_{j_1 \ldots j_{m+1}} = i\sigma(j_1 \ldots j_{m+1})(D_{j_1} v_{j_2 \ldots j_{m+1}}), \tag{2.4.9}$$

$$(\delta_0(D)v)_{j_1 \ldots j_{m-1}} = i\sum_{k=1}^{n} D_k v_{kj_1 \ldots j_{m-1}}. \tag{2.4.10}$$

To verify the Lopatinskiĭ condition for problem (2.4.6) we have to consider the next boundary value problem for a system of ordinary differential equations:

$$\delta_0(\xi', D_t)\, d_0(\xi', D_t)\, v(t) = 0, \tag{2.4.11}$$

$$v(0) = v_0, \tag{2.4.12}$$

where $D_t = -i d/dt$; and to prove that this problem has a unique solution in $\mathcal{N}_+$ for every $0 \neq \xi' \in \mathbf{R}^{n-1}$ and every tensor $v_0 \in S^m(\mathbf{R}^n)$. Here $\mathcal{N}_+$ is the space of solutions, to system (2.4.11), which tend to zero as $t \to \infty$.

Since the equation $\det(\delta_0(\xi', \lambda)d_0(\xi', \lambda)) = 0$ has real coefficients and has not a real root for $\xi' \neq 0$ as we have seen above, the space $\mathcal{N}$ of all solutions to system (2.4.11) can be represented as the direct sum: $\mathcal{N} = \mathcal{N}_+ \bigoplus \mathcal{N}_-$ where $\mathcal{N}_-$ is the space of solutions tending to zero as $t \to -\infty$. Moreover, $\dim \mathcal{N}_+ = \dim \mathcal{N}_- = \dim S^m(\mathbf{R}^n)$. Consequently, to verify the Lopatinskiĭ condition it is sufficient to show that the homogeneous problem

$$\delta_0(\xi', D_t)\, d_0(\xi', D_t)\, v(t) = 0, \quad v(0) = 0 \tag{2.4.13}$$

has only zero solution in the space $\mathcal{N}_+$. Before proving this, we will establish a Green formula.

Let $u(t)$ and $v(t)$ be symmetric tensors, on $\mathbf{R}^n$ of degree $m+1$ and m respectively, which depend smoothly on $t \in [0, \infty)$ and decrease rapidly together with all their derivatives as $t \to 0$. If $v(0) = 0$ then

$$\int_0^\infty \langle \delta_0(\xi', D_t)u, v \rangle\, dt = -\int_0^\infty \langle u, d_0(\xi', D_t)v \rangle\, dt. \tag{2.4.14}$$

The inner product is understood here according to definition (2.3.1) for $g_{ij} = \delta_{ij}$. Indeed,

$$\int_0^\infty \langle \delta_0(\xi', D_t)u, v \rangle\, dt = i \int_0^\infty \left(D_t u_{nj_1 \ldots j_m} + \sum_{k=1}^{n-1} \xi'_k u_{kj_1 \ldots j_m} \right) \overline{v}^{j_1 \ldots j_m}\, dt =$$

$$= i \int_0^\infty \left[u^{nj_1 \ldots j_m} \overline{(D_t v_{j_1 \ldots j_m})} + \sum_{k=1}^{n-1} \xi'_k u_{kj_1 \ldots j_m} \overline{v}^{j_1 \ldots j_m} \right] dt$$

(the bar means complex conjugating). Putting $\xi = (\xi'_1, \ldots, \xi'_{n-1}, D_t)$, we can rewrite this equality as:

$$\int_0^\infty \langle \delta_0(\xi', D_t)u, v \rangle\, dt = i \int_0^\infty u^{j_1 \ldots j_{m+1}} \overline{\xi_{j_1} v_{j_2 \ldots j_{m+1}}}\, dt. \tag{2.4.15}$$

By (2.4.9), we have $(d_0(\xi', D_t)v)_{j_1 \ldots j_{m+1}} = \sigma(j_1 \ldots j_{m+1})(\xi_{j_1} v_{j_2 \ldots j_{m+1}})$, where $\sigma(j_1 \ldots j_{m+1})$ is the symmetrization with respect to indices $j_1, \ldots, j_{m+1}$. Consequently, $\langle u, d_0(\xi', D_t)v \rangle = -iu^{j_1 \ldots j_{m+1}} \overline{\xi_{j_1} v_{j_2 \ldots j_{m+1}}}$. Comparing the last relation with (2.4.15), we arrive at (2.4.14).

Let $v(t) \in \mathcal{N}_+$ be a solution to problem (2.4.12). Putting $u(t) = d_0(\xi', D_t)v(t)$ in (2.4.13), we obtain

$$d_0(\xi', D_t)v(t) = 0. \tag{2.4.16}$$

Let us now prove that (2.4.16) and the initial condition $v(0) = 0$ imply that $v(t) \equiv 0$. Definition (2.4.9) for the operator $d_0(\xi)$ can be rewritten as

$$(d_0(\xi)v)_{j_1 \ldots j_{m+1}} = \frac{i}{m+1} \sum_{k=1}^{m+1} \xi_{j_k} v_{j_1 \ldots \widehat{j_k} \ldots j_{m+1}},$$

where the symbol $\wedge$ posed over j_k designates that this index is omitted. Putting $\xi = (\xi', D_t)$, $j_{m+1} = n$ in the last equality and taking (2.4.16) into account, we obtain

$$(d_0(\xi', D_t)v)_{nj_1 \ldots j_m} = \frac{i}{m+1} \left[(l+1) D_t v_{j_1 \ldots j_m} + \sum_{j_k \neq n} \xi_{j_k} v_{nj_1 \ldots \widehat{j_k} \ldots j_m} \right] = 0. \tag{2.4.17}$$

Here $l = l(j_1, \ldots, j_m)$ is the number of occurrences of the index n in $(j_1, \ldots, j_m)$. Thus the field $v(t)$ satisfies the homogeneous system (2.4.17) which is resolved with respect to derivatives. The last claim, together with the initial condition $v(0) = 0$, implies that $v(t) \equiv 0$. Ellipticity of problem (2.4.6) is proved.

For a field $u \in C^\infty(S^m \tau'_M)$ and a geodesic $\gamma : (a, b) \to M$, the following equality is valid:

$$\frac{d}{dt} \left[u_{i_1 \ldots i_m}(\gamma(t)) \dot\gamma^{i_1}(t) \ldots \dot\gamma^{i_m}(t) \right] = (du)_{i_1 \ldots i_{m+1}}(\gamma(t)) \dot\gamma^{i_1}(t) \ldots \dot\gamma^{i_{m+1}}(t). \tag{2.4.18}$$

It can be easily proved with the help of the operator $D/dt = \dot\gamma^i \nabla_i$ of total differentiation along γ. Indeed, using the equality $D\dot\gamma/dt = 0$, we obtain

$$\frac{d}{dt} \left(u_{i_1 \ldots i_m} \dot\gamma^{i_1} \ldots \dot\gamma^{i_m} \right) = \frac{D}{dt} \left(u_{i_1 \ldots i_m} \dot\gamma^{i_1} \ldots \dot\gamma^{i_m} \right) = \left(\frac{Du}{dt} \right)_{i_1 \ldots i_m} \dot\gamma^{i_1} \ldots \dot\gamma^{i_m} =$$

$$= u_{i_1 \ldots i_m \, ; \, j} \dot\gamma^j \dot\gamma^{i_1} \ldots \dot\gamma^{i_m} = (du)_{i_1 \ldots i_{m+1}} \dot\gamma^{i_1} \ldots \dot\gamma^{i_{m+1}}.$$

Let us prove that problem (2.4.6) has the trivial kernel; i.e., that the homogeneous problem $\delta du = 0$, $u|_{\partial M} = 0$ has only zero solution. By ellipticity, we can assume the field u to be smooth. Putting $v = du$ and $D = M$ in the Green formula (2.4.2), we obtain $du = 0$. Let $x_0 \in M \setminus \partial M$, and x_1 be a point in the boundary ∂M which is nearest to x_0. There exists a geodesic $\gamma : [-1, 0] \to M$ such that $\gamma(-1) = x_1$ and $\gamma(0) = x_0$. For a vector $\xi \in T_{x_0} M$, let γ_ξ be the geodesic defined by the initial conditions $\gamma_\xi(0) = x_0$, $\dot\gamma_\xi(0) = \xi$. If ξ is sufficiently close to $\dot\gamma(0)$, then γ_ξ intersects ∂M for some $t_0 = t_0(\xi) < 0$. Using (2.4.18), we obtain

$$u_{i_1 \ldots i_m}(x_0) \xi^{i_1} \ldots \xi^{i_m} = u_{i_1 \ldots i_m}(\gamma_\xi(t_0)) \dot\gamma_\xi^{i_1}(t_0) \ldots \dot\gamma_\xi^{i_m}(t_0) + \int_{t_0}^{0} (du)_{i_1 \ldots i_{m+1}}(\gamma_\xi(t)) \dot\gamma_\xi^{i_1}(t) \ldots \dot\gamma_\xi^{i_{m+1}}(t) \, dt = 0.$$

Since the last equality is valid for all ξ in a neighborhood of the vector $\dot\gamma(0)$ in $T_{x_0} M$, it implies that $u(x_0) = 0$. This means that $u \equiv 0$ because x_0 is arbitrary.

Let us prove that problem (2.4.6) has the trivial cokernel. Let a field $f \in C^\infty(S^m \tau'_M)$ be orthogonal to the range of the operator of the boundary value problem:

$$\int_M \langle f, \delta du \rangle \, dV^n = 0 \tag{2.4.19}$$

for every field $u \in C^\infty(S^m \tau'_M)$ satisfying the boundary condition

$$u|_{\partial M} = 0. \tag{2.4.20}$$

We have to show that $f = 0$. We first take u such that $\mathrm{supp}\, u \subset M \setminus \partial M$. From (2.4.19) with the help of the Green formula, we obtain

$$0 = \int_M \langle f, \delta du \rangle \, dV^n = -\int_M \langle df, du \rangle \, dV^n = \int_M \langle \delta d f, u \rangle \, dV^n.$$

Since $u \in C_0^\infty(S^m \tau'_M)$ is arbitrary, the last equality implies that

$$\delta d f = 0. \tag{2.4.21}$$

Now let $v \in C^\infty(S^m \tau'_M|_{\partial M})$ be arbitrary. One can easily see that there exists $u \in C^\infty(S^m \tau'_M)$ such that

$$u|_{\partial M} = 0, \qquad j_\nu du|_{\partial M} = v. \tag{2.4.22}$$

From (2.4.19), (2.4.21) and (2.4.22) with the help of the Green formula, we obtain

$$0 = \int\limits_M \langle f, \delta du \rangle \, dV^n = -\int\limits_M \langle d\,f, du \rangle \, dV^n + \int\limits_{\partial M} \langle f, j_\nu du \rangle \, dV^{n-1} =$$

$$= \int\limits_M \langle \delta d\, f, u \rangle \, dV^n + \int\limits_{\partial M} \langle f, v \rangle \, dV^{n-1} = \int\limits_{\partial M} \langle f, v \rangle \, dV^{n-1}.$$

Thus $\int_{\partial M} \langle f, v \rangle \, dV^{n-1} = 0$ for every $v \in C^\infty(S^m \tau'_M|_{\partial M})$ and, consequently, $f|_{\partial M} = 0$. As we know, the last equality and (2.4.21) imply that $f = 0$. The theorem is proved.

Proof of Lemma 4.2.3. For a symmetric tensor u of degree m, we obtain

$$(j_\xi i_\xi u)_{i_1 \ldots i_m} = \xi^{i_{m+1}} \sigma(i_1 \ldots i_{m+1})(u_{i_1 \ldots i_m} \xi_{i_{m+1}})$$

where $\sigma(i_1 \ldots i_{m+1})$ is the symmetrization in the indices $i_1 \ldots i_{m+1}$. Using the symmetry of u, we transform the right-hand side of the last equality as follows:

$$(j_\xi i_\xi u)_{i_1 \ldots i_m} = \frac{1}{m+1} \xi^{i_{m+1}} \sigma(i_1 \ldots i_m)(u_{i_1 \ldots i_m} \xi_{i_{m+1}} + m u_{i_2 \ldots i_{m+1}} \xi_{i_1}) =$$

$$= \frac{1}{m+1} \sigma(i_1 \ldots i_m)(u_{i_1 \ldots i_m} \xi_{i_{m+1}} \xi^{i_{m+1}} + m \xi_{i_1} u_{i_2 \ldots i_{m+1}} \xi^{i_{m+1}}) =$$

$$= \left(\frac{1}{m+1} |\xi|^2 u + \frac{m}{m+1} i_\xi j_\xi u \right)_{i_1 \ldots i_m}.$$

The lemma is proved.

2.5   Semibasic tensor fields

Given a manifold M, by $TM = \{(x, \xi) \mid x \in M, \xi \in T_x M\}$ we denote the set of all tangent vectors. Define the projection $p : TM \to M$, $p(x, \xi) = x$. The set TM is furnished by the structure of smooth manifold as follows. Let $(x^1, \ldots, x^n)$ be a local coordinate system on M with the domain $U \subset M$. For every point $(x, \xi) \in p^{-1}(U)$, there is the unique representation $\xi = \xi^i \frac{\partial}{\partial x^i}$. By the definition the set of functions $(x^1 \circ p, \ldots, x^n \circ p, \xi^1, \ldots, \xi^n)$ is the local coordinate system on TM with the domain $p^{-1}(U) \subset TM$. The family of such coordinate systems constitute the smooth atlas on TM. The triple $\tau_M = (TM, p, M)$ is called the *tangent bundle* of the manifold M.

Given a local coordinate system $(x^1, \ldots, x^n)$ on M with the domain $U \subset M$, we will use the brief notation x^i instead of $x^i \circ p$, hoping that it will not lead to misunderstanding. The coordinate system $(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ on TM with the domain $p^{-1}(U)$ will be called *associated* with the system $(x^1, \ldots, x^n)$. From now on we will use only such coordinate systems on TM. If $(x'^1, \ldots, x'^n)$ is another coordinate system defined in a domain $U' \subset M$, then in $p^{-1}(U \cap U')$ the associated coordinates are related by the transformation formulas

$$x'^i = x'^i(x^1, \ldots, x^n); \quad \xi'^i = \frac{\partial x'^i}{\partial x^j} \xi^j. \tag{2.5.1}$$

Unlike the case of general coordinates, these formulas have the next peculiarity: the first n transformation functions are independent of ξ^i while the last n functions depend linearly on these variables. This peculiarity is the base of all further constructions in the current section.

The algebra of tensor fields of the manifold TM is generated locally by the coordinate fields $\partial/\partial x^i$, $\partial/\partial \xi^i$, dx^i, $d\xi^i$. Differentiating (2.5.1), we obtain the next rules for transforming the fields with respect to change of associated coordinates:

$$\frac{\partial}{\partial \xi^i} = \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial \xi'^j}, \qquad dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j, \tag{2.5.2}$$

$$\frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i}\frac{\partial}{\partial x'^j} + \frac{\partial^2 x'^j}{\partial x^i \partial x^k}\xi^k\frac{\partial}{\partial \xi'^j}, \quad d\xi'^i = \frac{\partial^2 x'^i}{\partial x^j \partial x^k}\xi^k dx^j + \frac{\partial x'^i}{\partial x^j}d\xi^j. \tag{2.5.3}$$

We note that formulas (2.5.2) contain only the first-order derivatives of the transformation functions and take the observation as the basis for the next definition.

A tensor field $u \in C^\infty(\tau_s^r(TM))$ of degree (r, s) on the manifold TM is called *semibasic* if in an associated coordinate system it can be represented as:

$$u = u_{j_1 \ldots j_s}^{i_1 \ldots i_r}\frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \tag{2.5.4}$$

with coefficients $u_{j_1 \ldots j_s}^{i_1 \ldots i_r} \in C^\infty(p^{-1}(U))$ (where $U \subset M$ is the domain of the corresponding coordinate system on M) that are called the *coordinates* (or *components*) of the field u. Assuming the choice of the coordinate system to be clear from the context (or arbitrary), we will abbreviate equality (2.5.4) to the next one:

$$u = (u_{j_1 \ldots j_s}^{i_1 \ldots i_r}). \tag{2.5.5}$$

It follows from (2.5.2) that, under a change of an associated coordinate system, the components of a semibasic tensor field are transformed by the formulas

$$u_{j_1 \ldots j_s}'^{i_1 \ldots i_r} = \frac{\partial x'^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial x'^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{l_s}}{\partial x'^{j_s}}u_{l_1 \ldots l_s}^{k_1 \ldots k_r} \tag{2.5.6}$$

which are identical in form with formulas (2.1.5) for transforming components of an ordinary tensor field on M. The set of all semibasic tensor fields of degree (r, s) will be denoted by $C^\infty(\beta_s^r M)$. Note that $C^\infty(\beta_0^0 M) = C^\infty(TM)$, i.e., semibasic tensor fields of degree $(0, 0)$ are just smooth functions on TM. The elements of $C^\infty(\beta_0^1 M)$ are called *semibasic vector fields*, and the elements of $C^\infty(\beta_1^0 M)$ are called *semibasic covector fields*.

Formula (2.5.6) establishes a formal analogy between ordinary tensor fields and semibasic tensor fields. Using the analogy, we introduce some algebraic and differential operations on semibasic tensor fields.

The set $C^\infty(\beta_s^r M)$ is a $C^\infty(TM)$-module, i.e., the semibasic tensor fields of the same degree can be summed and multiplied by functions $\varphi(x, \xi)$ depending smoothly on $(x, \xi) \in TM$.

For $u \in C^\infty(\beta_s^r M)$ and $v \in C^\infty(\beta_{s'}^{r'} M)$ the *tensor product* $u \otimes v \in C^\infty(\beta_{s+s'}^{r+r'} M)$ is defined in coordinate form by formula (2.1.8). With the help of (2.5.6) by standard arguments, one proves correctness of this definition, i.e., that the field $u \otimes v$ is independent of the choice of an associated coordinate system participating in the definition. The so-obtained operation turns $C^\infty(\beta_*^* M) = \bigoplus_{r,s=0}^\infty C^\infty(\beta_s^r M)$ into a bigraded $C^\infty(TM)$-algebra. This algebra is generated locally by the coordinate semibasic fields $\partial/\partial \xi^i$ and dx^i.

The *operations of transposition of upper and lower indices* are defined by formulas (2.1.6), and the *contraction operators* $C_l^k : C^\infty(\beta_s^r M) \to C^\infty(\beta_{s-1}^{r-1} M)$ are defined by (2.1.7). With the help of (2.5.6), one verifies correctness of these definitions.

Tensor fields on M can be identified with the semibasic tensor fields on TM whose components are independent of the second argument ξ. Let us call such the fields *basic fields*. Formula (2.5.6) implies that this property is independent of choice of associated coordinates. Thus we obtain the canonical imbedding

$$\kappa : C^\infty(\tau_s^r M) \subset C^\infty(\beta_s^r M) \tag{2.5.7}$$

which is compatible with all algebraic operations introduced above. Note that $\kappa(\partial/\partial x^i) = \partial/\partial \xi^i$ and $\kappa(dx^i) = dx^i$.

Given $u \in C^\infty(\beta_s^r M)$, it follows from (2.5.6) that the set of the functions

$$\overset{v}{\nabla}_k u_{j_1 \ldots j_s}^{i_1 \ldots i_r} = \frac{\partial}{\partial \xi^k}u_{j_1 \ldots j_s}^{i_1 \ldots i_r} \tag{2.5.8}$$

is the set of components of a semibasic field of degree $(r, s + 1)$. The equality

$$\overset{v}{\nabla} u = \overset{v}{\nabla}_k u_{j_1 \ldots j_s}^{i_1 \ldots i_r}\frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^k \tag{2.5.9}$$

defines correctly the differential operator $\overset{v}{\nabla}: C^\infty(\beta_s^r M) \to C^\infty(\beta_{s+1}^r M)$ which will be called the *vertical covariant derivative*. One can verify directly that $\overset{v}{\nabla}$ commutes with the contraction operators and is related to the tensor product by the equality

$$\overset{v}{\nabla}(u \otimes v) = \rho_{s+1}(\overset{v}{\nabla} u \otimes v) + u \otimes \overset{v}{\nabla} v \tag{2.5.10}$$

for $u \in C^\infty(\beta_s^r M)$, where ρ_{s+1} is the same as in (2.2.5).

2.6   The horizontal covariant derivative

In this section M is a Riemannian manifold with metric tensor g.

The *geodesic flow* is the local one-parameter group $G^t : TM \to TM$ of diffeomophisms of the tangent manifold TM which are defined by the equality $G^t(x,\xi) = (\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))$ (recall that $\gamma_{x,\xi}(t)$ is the geodesic starting from x in the direction ξ). The vector field H on the manifold TM generating the flow is expressed in associated coordinates as follows:

$$H = \xi^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} \xi^j \xi^k \frac{\partial}{\partial \xi^i}. \tag{2.6.1}$$

Indeed, let $u \in C^\infty(TM)$. Using the equation (2.3.5) of geodesics, we see that

$$\left. \frac{d(u \circ G^t)}{dt} \right|_{t=0} = Hu = \xi^i \left(\frac{\partial u}{\partial x^i} - \Gamma^p_{iq} \xi^q \frac{\partial u}{\partial \xi^p} \right).$$

The first factor on the right-hand side of this equality is the component of the semibasic vector field $\xi = (\xi^i)$. Invariance of the function Hu suggests that the second factor on the right-hand side is also the component of some semibasic covector field. This observation we use as a basis for the next definition.

The *horizontal covariant derivative* of a function $u \in C^\infty(TM) = C^\infty(\beta^0_0 M)$ is the semibasic covector field $\overset{h}{\nabla} u \in C^\infty(\beta^0_1 M)$ given in an associated coordinate system by the equalities

$$\overset{h}{\nabla} u = (\overset{h}{\nabla}_k u) dx^k, \qquad \overset{h}{\nabla}_k u = \frac{\partial u}{\partial x^k} - \Gamma^p_{kq} \xi^q \frac{\partial u}{\partial \xi^p}. \tag{2.6.2}$$

To show correctness of the definition we have to prove that, under a change of an associated coordinate system, the functions (2.6.2) are transformed by formulas (2.5.6) for $r = 0$ and $s = 1$. Using (2.2.7), (2.5.2) and (2.5.3), we obtain

$$\overset{h}{\nabla}'_k u = \left(\frac{\partial}{\partial x'^k} - \Gamma'^p_{kq} \xi'^q \frac{\partial}{\partial \xi'^p} \right) u =$$

$$= \left[\frac{\partial x^\alpha}{\partial x'^k} \frac{\partial}{\partial x^\alpha} + \frac{\partial^2 x^\alpha}{\partial x'^k \partial x'^i} \xi'^i \frac{\partial}{\partial \xi^\alpha} - \left(\frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^k} \frac{\partial x^\gamma}{\partial x'^q} \Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^k \partial x'^q} \right) \xi'^q \frac{\partial x^\varepsilon}{\partial x'^p} \frac{\partial}{\partial \xi^\varepsilon} \right] u.$$

Changing the notation of summation indices, we rewrite this equality as follows:

$$\overset{h}{\nabla}'_k u = \frac{\partial x^\alpha}{\partial x'^k} \left[\frac{\partial}{\partial x^\alpha} - \left(\frac{\partial x'^p}{\partial x^\beta} \frac{\partial x^\varepsilon}{\partial x'^p} \right) \frac{\partial x^\gamma}{\partial x'^q} \xi'^q \Gamma^\beta_{\alpha\gamma} \frac{\partial}{\partial \xi^\varepsilon} \right] u + \frac{\partial^2 x^\alpha}{\partial x'^k \partial x'^q} \xi'^q \left[\frac{\partial}{\partial \xi^\alpha} - \left(\frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\varepsilon}{\partial x'^p} \right) \frac{\partial}{\partial \xi^\varepsilon} \right] u.$$

Using (2.5.1) and taking it into account that the matrices $\partial x'/\partial x$ and $\partial x/\partial x'$ are inverse to one other, we finally obtain

$$\overset{h}{\nabla}'_k u = \frac{\partial x^\alpha}{\partial x'^k} \left(\frac{\partial u}{\partial x^\alpha} - \Gamma^\beta_{\alpha\gamma} \xi^\gamma \frac{\partial u}{\partial \xi^\beta} \right) = \frac{\partial x^\alpha}{\partial x'^k} \overset{h}{\nabla}_\alpha u.$$

Thus correctness of the definition of the operator $\overset{h}{\nabla} : C^\infty(\beta^0_0 M) \to C^\infty(\beta^0_1 M)$ is proved.

By analogy with Theorem 2.2.1 we formulate the next

Theorem 2.6.1 *Let M be a Riemannian manifold. For all integers r and s, there exist uniquely determined $\mathbf{R}$-linear operators*

$$\overset{h}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M) \tag{2.6.3}$$

such that

(1) *on basic tensor fields, $\overset{h}{\nabla}$ coincides with the operator ∇ of covariant differentiation with respect to the Levi-Civita connection, i.e., $\overset{h}{\nabla}(\kappa u) = \kappa(\nabla u)$ for $u \in C^\infty(\tau^r_s M)$, where κ is imbedding (2.5.7);*

(2) *on $C^\infty(\beta^0_0 M)$, $\overset{h}{\nabla}$ coincides with operator (2.6.2);*

(3) *$\overset{h}{\nabla}$ commutes with the contraction operators C^k_l for $1 \le k \le r$, $1 \le l \le s$;*

(4) *$\overset{h}{\nabla}$ is related to the tensor product by the equality*

$$\overset{h}{\nabla}(u \otimes v) = \rho_{s+1}(\overset{h}{\nabla} u \otimes v) + u \otimes \overset{h}{\nabla} v \tag{2.6.4}$$

for $u \in C^\infty(\beta_s^r M)$, where ρ_{s+1} is the same as in (2.2.5).

In an associated coordinate system, for $u \in C^\infty(\beta_s^r M)$, the next local representation is valid:

$$\overset{h}{\nabla} u = \overset{h}{\nabla}_k u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^k, \qquad (2.6.5)$$

where

$$\overset{h}{\nabla}_k u^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial}{\partial x^k} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} - \Gamma^p_{kq} \xi^q \frac{\partial}{\partial \xi^p} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} + \sum_{\alpha=1}^{r} \Gamma^{i_\alpha}_{kp} u^{i_1 \ldots i_{\alpha-1} p i_{\alpha+1} \ldots i_r}_{j_1 \ldots j_s} - \sum_{\alpha=1}^{s} \Gamma^p_{k j_\alpha} u^{i_1 \ldots i_r}_{j_1 \ldots j_{\alpha-1} p j_{\alpha+1} \ldots j_s}. \quad (2.6.6)$$

Pay attention to a formal analogy between the formulas (2.2.9) and (2.6.6): comparing with (2.2.9), the right-hand side of (2.6.6) contains one additional summand related to dependence of components of the field u on the coordinates ξ^i.

Proof. Let operators (2.6.3) satisfy conditions (1)–(4) of the theorem; we prove the validity of the local representation (2.6.5)–(2.6.6).

The tensor fields

$$\frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} = \kappa \left(\frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \right) \qquad (2.6.7)$$

are basic. By the first condition of the theorem,

$$\overset{h}{\nabla} \left(\frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \right) =$$

$$= \sum_{\alpha=1}^{r} \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_{\alpha-1}}} \otimes \Gamma^p_{k i_\alpha} \frac{\partial}{\partial \xi^p} \otimes \frac{\partial}{\partial \xi^{i_{\alpha+1}}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^k -$$

$$- \sum_{\alpha=1}^{s} \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_{\alpha-1}} \otimes \Gamma^{j_\alpha}_{kp} dx^p \otimes dx^{j_{\alpha+1}} \otimes \ldots \otimes dx^{j_s} \otimes dx^k. \qquad (2.6.8)$$

Given $u \in C^\infty(\beta_s^r M)$, we apply the fourth condition of the theorem and obtain

$$\overset{h}{\nabla} u = \overset{h}{\nabla} \left(u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \right) =$$

$$= \rho_1 \left[\left(\overset{h}{\nabla} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \right) \otimes \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \right] +$$

$$+ u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \overset{h}{\nabla} \left(\frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \right), \qquad (2.6.9)$$

where the expression $\overset{h}{\nabla} u^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ denotes the result of applying $\overset{h}{\nabla}$ to the scalar function $u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in C^\infty(\beta_0^0 M)$. By the second condition of the theorem, this expression can be found by formula (2.6.2). Along the same lines by using (2.6.8), we transform equality (2.6.9) as follows:

$$\overset{h}{\nabla} u = \left(\frac{\partial}{\partial x^k} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} - \Gamma^p_{kq} \xi^q \frac{\partial}{\partial \xi^p} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \right) \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^k +$$

$$+ \sum_{\alpha=1}^{r} \Gamma^p_{k i_\alpha} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_{\alpha-1}}} \otimes \frac{\partial}{\partial \xi^p} \otimes \frac{\partial}{\partial \xi^{i_{\alpha+1}}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s} \otimes dx^k -$$

$$- \sum_{\alpha=1}^{s} \Gamma^{j_\alpha}_{kp} u^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_{\alpha-1}} \otimes dx^p \otimes dx^{j_{\alpha+1}} \otimes \ldots \otimes dx^{j_s} \otimes dx^k.$$

Changing the limits of summation over the indices i_α and p in the first sum of the right-hand side and changing the limits of summation over j_α and p in the second sum, we arrive at (2.6.5) and (2.6.6).

Conversely, let us define the operators $\overset{h}{\nabla}$ by formulas (2.6.5)–(2.6.6) in an associated coordinate system, with the help of arguments similar to those we have used just after definition (2.6.2), one can prove correctness of this definition. Thereafter validity of claims (1)–(4) of the theorem can easily be proved by a straightforward calculation in coordinate form. The theorem is proved.

Theorem 2.6.2 *The vertical and horizontal derivatives satisfy the next commutation relations:*

$$(\overset{v}{\nabla}_k\overset{v}{\nabla}_l - \overset{v}{\nabla}_l\overset{v}{\nabla}_k)u^{i_1\ldots i_r}_{j_1\ldots j_s} = 0, \tag{2.6.10}$$

$$(\overset{v}{\nabla}_k\overset{h}{\nabla}_l - \overset{h}{\nabla}_l\overset{v}{\nabla}_k)u^{i_1\ldots i_r}_{j_1\ldots j_s} = 0, \tag{2.6.11}$$

$$(\overset{h}{\nabla}_k\overset{h}{\nabla}_l - \overset{h}{\nabla}_l\overset{h}{\nabla}_k)u^{i_1\ldots i_r}_{j_1\ldots j_s} = -R^p_{qkl}\xi^q\overset{v}{\nabla}_p u^{i_1\ldots i_r}_{j_1\ldots j_s} + \sum_{\alpha=1}^{r} R^{i_\alpha}_{pkl} u^{i_1\ldots i_{\alpha-1}p i_{\alpha+1}\ldots i_r}_{j_1\ldots j_s} - \sum_{\alpha=1}^{s} R^p_{j_\alpha kl} u^{i_1\ldots i_r}_{j_1\ldots j_{\alpha-1}p j_{\alpha+1}\ldots j_s}. \tag{2.6.12}$$

We again pay attention to a formal analogy between the formulas (2.2.10) and (2.6.12).

Proof. Equality (2.6.10) is evident, since $\overset{v}{\nabla}_k = \partial/\partial\xi^k$. To prove (2.6.11) we differentiate equality (2.6.6) with respect to ξ^l:

$$\overset{v}{\nabla}_l\overset{h}{\nabla}_k u^{i_1\ldots i_r}_{j_1\ldots j_s} = \frac{\partial}{\partial x^k}\overset{v}{\nabla}_l u^{i_1\ldots i_r}_{j_1\ldots j_s} - \Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}\overset{v}{\nabla}_l u^{i_1\ldots i_r}_{j_1\ldots j_s} - \Gamma^p_{kl}\overset{v}{\nabla}_p u^{i_1\ldots i_r}_{j_1\ldots j_s} +$$

$$+\sum_{\alpha=1}^{r}\Gamma^{i_\alpha}_{kp}\overset{v}{\nabla}_l u^{i_1\ldots i_{\alpha-1}p i_{\alpha+1}\ldots i_r}_{j_1\ldots j_s} - \sum_{\alpha=1}^{s}\Gamma^p_{kj_\alpha}\overset{v}{\nabla}_l u^{i_1\ldots i_r}_{j_1\ldots j_{\alpha-1}p j_{\alpha+1}\ldots j_s}.$$

Including the third summand on the right-hand side into the last sum, we arrive at (2.6.11).

We will prove (2.6.12) only for $r = s = 0$. In other cases this formula is proved by similar but more cumbersome calculations. For $u \in C^\infty(\beta^0_0 M)$, we obtain

$$\overset{h}{\nabla}_k\overset{h}{\nabla}_l u = \left(\frac{\partial}{\partial x^k} - \Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}\right)\overset{h}{\nabla}_l u - \Gamma^p_{kl}\overset{h}{\nabla}_p u =$$

$$= \left(\frac{\partial}{\partial x^k} - \Gamma^p_{kq}\xi^q\frac{\partial}{\partial\xi^p}\right)\left(\frac{\partial u}{\partial x^l} - \Gamma^j_{lr}\xi^r\frac{\partial u}{\partial\xi^j}\right) - \Gamma^p_{kl}\left(\frac{\partial u}{\partial x^p} - \Gamma^j_{pq}\xi^q\frac{\partial u}{\partial\xi^j}\right).$$

After opening the parenthesis and changing notation in summation indices, this equality takes the form

$$\overset{h}{\nabla}_k\overset{h}{\nabla}_l u = \frac{\partial^2 u}{\partial x^k\partial x^l} - \Gamma^p_{lq}\xi^q\frac{\partial^2 u}{\partial x^k\partial\xi^p} - \Gamma^p_{kq}\xi^q\frac{\partial^2 u}{\partial x^l\partial\xi^p} + \Gamma^p_{kq}\Gamma^j_{lr}\xi^q\xi^r\frac{\partial^2 u}{\partial\xi^p\partial\xi^j} -$$

$$- \Gamma^p_{kl}\frac{\partial u}{\partial x^p} - \left(\frac{\partial\Gamma^p_{lq}}{\partial x^k} - \Gamma^j_{kq}\Gamma^p_{lj} - \Gamma^j_{kl}\Gamma^p_{jq}\right)\xi^q\overset{v}{\nabla}_p u. \tag{2.6.13}$$

Alternating (2.6.13) with respect to k and l, we come to

$$(\overset{h}{\nabla}_k\overset{h}{\nabla}_l - \overset{h}{\nabla}_l\overset{h}{\nabla}_k)u = -\left(\frac{\partial\Gamma^p_{lq}}{\partial x^k} - \frac{\partial\Gamma^p_{kq}}{\partial x^l} + \Gamma^j_{lq}\Gamma^p_{kj} - \Gamma^j_{kq}\Gamma^p_{lj}\right)\xi^q\overset{v}{\nabla}_p u.$$

By (2.2.8), the last equality coincides with (2.6.12) for $r = s = 0$. The theorem is proved.

Note that the next relations are valid:

$$\overset{v}{\nabla}_k g_{ij} = \overset{h}{\nabla}_k g^{ij} = 0, \quad \overset{v}{\nabla}_k\delta^i_j = \overset{h}{\nabla}_k\delta^i_j = 0, \quad \overset{h}{\nabla}_k\xi^i = 0, \quad \overset{v}{\nabla}_k\xi^i = \delta^i_k,$$

where δ^j_i is the Kronecker tensor: $\delta^j_i = 1$ for $i = j$, and $\delta^j_i = 0$ for $i \neq j$.

In what follows we will also use the notations: $\overset{v}{\nabla}^i = g^{ij}\overset{v}{\nabla}_j$, $\overset{h}{\nabla}^i = g^{ij}\overset{h}{\nabla}_j$.

Operator (2.6.1) can be extended to semibasic tensor fields of arbitrary degree

$$H : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_s M)$$

by the definition $H = \xi^i\overset{h}{\nabla}_i$.

2.7 Formulas of Gauss — Ostrogradskiĭ type for vertical and horizontal derivatives

Given a Riemannian manifold M, by $\Omega M = \{(x,\xi) \in TM \mid |\xi| = 1\}$ we denote the manifold of unit tangent vectors, and by $\Omega_x M = \Omega M \cap T_x M$, the unit sphere at a point $x \in M$. Since $T_x M$ is endowed with the structure of Euclidean vector space, $\Omega_x M$ is endowed with the corresponding volume form that will be denoted by $d\omega_x$.

Theorem 2.7.1 *Let M be a Riemannian manifold of dimension n and $u = u(x,\xi)$ be a semibasic vector field on TM positively homogeneous of degree λ in ξ: $u(x,t\xi) = t^\lambda u(x,\xi)$ $(t > 0)$. For every compact domain $G \subset M$ with a piecewise smooth boundary ∂G the next Gauss — Ostrogradskiĭ formulas are valid:*

$$\int\limits_G \int\limits_{\Omega_x M} \overset{v}{\nabla}_i u^i \, d\omega_x(\xi) \, dV^n(x) = (\lambda + n - 1) \int\limits_G \int\limits_{\Omega_x M} \langle u, \xi \rangle \, d\omega_x(\xi) \, dV^n(x), \qquad (2.7.1)$$

$$\int\limits_G \int\limits_{\Omega_x M} \overset{h}{\nabla}_i u^i \, d\omega_x(\xi) \, dV^n(x) = (-1)^n \int\limits_{\partial G} \int\limits_{\Omega_x M} \langle u, \nu \rangle \, d\omega_x(\xi) \, dV^{n-1}(x). \qquad (2.7.2)$$

Here $dV^n(x)$ and $dV^{n-1}(x)$ are the Riemannian volumes on M and ∂G respectively, and ν is the unit vector of the outer normal to ∂G.

Proof of formula (2.7.1). Fix a point $x \in M$ and choose local coordinates in a neighborhood of the point such that $g_{ij}(x) = \delta_{ij}$. For $\rho > 1$, introduce the notations $D_{x,\rho} = \{\xi \in T_x M \mid 1 \le |\xi| \le \rho\}$. Applying the classical Gauss — Ostrogradskiĭ formula, we can write

$$\int\limits_{D_{x,\rho}} \overset{v}{\nabla}_i u^i \, d\xi = \int\limits_{D_{x,\rho}} \frac{\partial u^i}{\partial \xi^i} \, d\xi = \rho^{n-1} \int\limits_{\Omega_x M} \langle \xi, u(x, \rho\xi) \rangle \, d\omega_x(\xi) - \int\limits_{\Omega_x M} \langle \xi, u(x,\xi) \rangle \, d\omega_x(\xi).$$

Using the homogeneity of u, we rewrite this equality in the form

$$\int\limits_1^\rho t^{\lambda+n-2} \, dt \int\limits_{\Omega_x M} \overset{v}{\nabla}_i u^i \, d\omega_x(\xi) = \rho^{\lambda+n-1} \int\limits_{\Omega_x M} \langle \xi, u(x,\xi) \rangle \, d\omega_x(\xi) - \int\limits_{\Omega_x M} \langle \xi, u(x,\xi) \rangle \, d\omega_x(\xi).$$

Differentiating this equality with respect to ρ and putting then $\rho = 1$, we obtain

$$\int\limits_{\Omega_x M} \overset{v}{\nabla}_i u^i \, d\omega_x(\xi) = (\lambda + n - 1) \int\limits_{\Omega_x M} \langle \xi, u(x,\xi) \rangle \, d\omega_x(\xi).$$

Multiplying the latter equality by $dV^n(x)$ and integrating over G, we get (2.7.1).

The rest of the section is devoted to the proof of formula (2.7.2).

There is the natural volume form dV^{2n} on the manifold TM which is defined by the equality

$$dV^{2n} = g \, d\xi \wedge dx = g \, d\xi^1 \wedge \ldots \wedge d\xi^n \wedge dx^1 \wedge \ldots \wedge dx^n \qquad (2.7.3)$$

in associated coordinates. From now on in this section $g = \det(g_{jk})$, and d is the exterior derivative.

In the domain of an associated coordinate system we introduce $(2n-1)$-forms:

$$\overset{h}{\omega}_i = g\Big[(-1)^{n+i-1} d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n + \sum_{j=1}^n (-1)^j \Gamma_{ip}^j \xi^p d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\Big]. \quad (2.7.4)$$

We recall that the symbol $\wedge$ over a factor designates that the factor is omitted.

Lemma 2.7.2 *The forms $\overset{h}{\omega}_i$ have the following properties:*

(1) under a change of an associated coordinate system the family $(\overset{h}{\omega}_i)$ transforms according to the same rule as components of a semibasic covector field; consequently, for every semibasic vector field $u = (u^i)$, the form $u^i \overset{h}{\omega}_i$ is independent of the choice of an associated coordinate system and is defined globally on TM;

(2) For a semibasic vector field $u = (u^i)$ the next equality is valid:

$$d(u^i \overset{h}{\omega}_i) = \overset{h}{\nabla}_i u^i \, dV^{2n}. \qquad (2.7.5)$$

Proof. First of all we note that for coincidence of two $(d-1)$-forms α and β defined on a d-dimensional manifold X it is sufficient that the equalities $\alpha \wedge dx^i = \beta \wedge dx^i$ $(i = 1, \ldots, d)$ are valid for any local coordinate system $(x^1, \ldots, x^d)$ on X.

The first claim of the lemma says that, in the intersection of domains of two associated coordinate systems, the next relations are valid:

$$\overset{h}{\omega}{}'_i \wedge dx'^k = \frac{\partial x^j}{\partial x'^i} \overset{h}{\omega}_j \wedge dx'^k, \quad \overset{h}{\omega}{}'_i \wedge d\xi'^k = \frac{\partial x^j}{\partial x'^i} \overset{h}{\omega}_j \wedge d\xi'^k. \tag{2.7.6}$$

By (2.7.3) and (2.7.4), the left-hand side of the first of equalities (2.7.6) is equal to $-\delta_i^k dV^{2n}$. We find the right-hand side of this equality with the help of (2.5.2):

$$\frac{\partial x^j}{\partial x'^i} \overset{h}{\omega}_j \wedge dx'^k = (-1)^{n+j-1} \frac{\partial x^j}{\partial x'^i} \frac{\partial x'^k}{\partial x^l} g \, d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \wedge dx^l =$$

$$= -\frac{\partial x^j}{\partial x'^i} \frac{\partial x'^k}{\partial x^l} \delta_j^l \, dV^{2n} = -\delta_i^k dV^{2n}.$$

By (2.7.4), the left-hand side of the second of formulas (2.7.6) is equal to

$$\overset{h}{\omega}{}'_i \wedge d\xi'^k = \Gamma'^k_{iq} \xi'^q \, dV^{2n} = \frac{\partial x'^q}{\partial x^p} \Gamma'^k_{iq} \xi^p \, dV^{2n}. \tag{2.7.7}$$

We calculate the right-hand side of this equality with the help of (2.5.3):

$$\frac{\partial x^j}{\partial x'^i} \overset{h}{\omega}_j \wedge d\xi'^k = g \frac{\partial x^j}{\partial x'^i} \left[(-1)^{n+j-1} d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \right. +$$

$$+ \sum_{l=1}^n (-1)^l \, \Gamma^l_{jp} \xi^p \, d\xi^1 \wedge \ldots \wedge \widehat{d\xi^l} \wedge \ldots \wedge d\xi^n \wedge dx \Bigg] \wedge \left(\frac{\partial^2 x'^k}{\partial x^q \partial x^r} \xi^q \, dx^r + \frac{\partial x'^k}{\partial x^q} d\xi^q \right) =$$

$$= g \frac{\partial x^j}{\partial x'^i} \left[(-1)^{n+j-1} \frac{\partial^2 x'^k}{\partial x^q \partial x^r} \xi^q \, d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^j} \wedge \ldots \wedge dx^n \wedge dx^r \right. +$$

$$+ \sum_{l=1}^n (-1)^l \frac{\partial x'^k}{\partial x^q} \Gamma^l_{jp} \, \xi^p \, d\xi^1 \wedge \ldots \wedge \widehat{d\xi^l} \wedge \ldots \wedge d\xi^n \wedge dx \wedge d\xi^q \Bigg] =$$

$$= \frac{\partial x^j}{\partial x'^i} \left[-\frac{\partial^2 x'^k}{\partial x^q \partial x^r} \xi^q \delta_j^r + \sum_{l=1}^n \frac{\partial x'^k}{\partial x^q} \Gamma^l_{jp} \xi^p \delta_l^q \right] dV^{2n}.$$

After summing over r and l, we obtain

$$\frac{\partial x^j}{\partial x'^i} \overset{h}{\omega}_j \wedge dx'^k = \frac{\partial x^j}{\partial x'^i} \left(-\frac{\partial^2 x'^k}{\partial x^j \partial x^p} + \frac{\partial x'^k}{\partial x^l} \Gamma^l_{jp} \right) \xi^p \, dV^{2n}.$$

Comparing the last relation with (2.7.7), we see that to prove the second of the equalities (2.7.6) it is sufficient to show that

$$\frac{\partial x'^q}{\partial x^p} \Gamma'^k_{iq} = \frac{\partial x^j}{\partial x'^i} \left(-\frac{\partial^2 x'^k}{\partial x^j \partial x^p} + \frac{\partial x'^k}{\partial x^l} \Gamma^l_{jp} \right). \tag{2.7.8}$$

These relations are equivalent to formulas (2.2.7) of transformation of the Christoffel symbols, as one can verify by multiplying (2.7.8) by $\partial x'^i / \partial x^r$ and summing over i. Thus the first claim of the lemma is proved.

We find the differential of the form $\overset{h}{\omega}_i$. From (2.7.4), we obtain

$$d\overset{h}{\omega}_i = (-1)^{n+i-1} \frac{\partial g}{\partial x^k} \, dx^k \wedge d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n +$$

$$+ g \sum_{j=1}^n (-1)^j \Gamma^j_{ip} \, d\xi^p \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx = \left(\frac{\partial g}{\partial x^i} - g\Gamma^j_{ij} \right) d\xi \wedge dx.$$

From this, using the relation

$$\Gamma^j_{ij} = \frac{1}{2} \frac{\partial}{\partial x^i} (\ln g), \tag{2.7.9}$$

which follows from (2.3.4), we conclude that

$$d\overset{h}{\omega}_i = \Gamma^j_{ij}\, dV^{2n}. \tag{2.7.10}$$

Let us now prove the second claim of the lemma. Let $u = (u^i)$ be a semibasic vector field. With the help of (2.7.10), we derive

$$d(u^i \overset{h}{\omega}_i) = du^i \wedge \overset{h}{\omega}_i + u^i d\overset{h}{\omega}_i =$$

$$= g\sum_{i=1}^{n}\left(\frac{\partial u^i}{\partial x^k}dx^k + \frac{\partial u^i}{\partial \xi^k}d\xi^k\right)\wedge\left[(-1)^{n+i-1}d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n +\right.$$

$$\left.+\sum_{j=1}^{n}(-1)^j\Gamma^j_{ip}\,\xi^p \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\right] + u^i\Gamma^j_{ij}\,dV^{2n} =$$

$$= g\left[\sum_{i=1}^{n}(-1)^{n+i-1}\frac{\partial u^i}{\partial x^k}\,dx^k \wedge d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n +\right.$$

$$\left.+\sum_{j=1}^{n}(-1)^j\Gamma^j_{ip}\xi^p\frac{\partial u^i}{\partial \xi^k}\,d\xi^k \wedge d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\right] + u^i\Gamma^j_{ij}\,dV^{2n} =$$

$$= \left(\frac{\partial u^i}{\partial x^i} + \Gamma^j_{ij}u^i - \Gamma^k_{ip}\xi^p\frac{\partial u^i}{\partial \xi^k}\right)dV^{2n} = \overset{h}{\nabla}_i u^i\,dV^{2n}.$$

The lemma is proved.

Applying the Stokes theorem, from (2.7.5), we obtain the next Gauss-Ostrogradskiĭ formula for the horizontal divergence:

$$\int_D \overset{h}{\nabla}_i u^i\, dV^{2n} = \int_{\partial D} u^i \overset{h}{\omega}_i, \tag{2.7.11}$$

which is valid for a semibasic vector field $u = (u^i)$ and a compact domain $D \subset TM$ with the piecewise smooth boundary ∂D.

We will need the next simple assertion whose proof is omitted due to its clarity.

Lemma 2.7.3 *Let α be a $(d-1)$-form on a d-dimensional manifold X, and $Y \subset X$ be a submanifold of codimension one which is determined by an equation $f(x) = 0$ such that $d\,f(x) \neq 0$ for $x \in Y$. The restriction of the form α to the submanifold Y equals zero if and only if $(\alpha \wedge d\,f)(x) = 0$ for all $x \in Y$.*

Formula (2.7.11) can be simplified essentially for some particular type of a domain D which is of import for us. Let G be a compact domain in M with piecewise smooth boundary ∂G. For $1 < \rho$, by $T_{1,\rho}G$ we denote the domain in TM that is defined by the equality $T_{1,\rho}G = \{(x,\xi) \in TM \mid x \in G,\ 1 \leq |\xi| \leq \rho\}$. The boundary of the domain is the union of three piecewise smooth manifolds:

$$\partial(T_{1,\rho}G) = \Omega_\rho G - \Omega G + T_{1,\rho}(\partial G), \tag{2.7.12}$$

where

$$\Omega G = \{(x,\xi) \in \Omega M \mid x \in G\}, \quad \Omega_\rho G = \{(x,\xi) \in TM \mid x \in G,\ |\xi| = \rho\},$$

$$T_{1,\rho}(\partial G) = \{(x,\xi) \in TM \mid x \in \partial G,\ 1 \leq |\xi| \leq \rho\}.$$

We have the diffeomorphism

$$\mu : \Omega G \to \Omega_\rho G, \quad \mu(x,\xi) = (x,\rho\xi). \tag{2.7.13}$$

The second summand on the right-hand side of (2.7.12) is furnished with the minus sign to emphasize that it enters into $\partial(T_{1,\rho}G)$ with the orientation opposite to that induced by the diffeomorphism μ.

Let us show that the restriction to $\Omega_\rho G$ of each of the forms $\overset{h}{\omega}_i$ is equal to zero. By Lemma 2.7.3, to this end it is sufficient to verify the equality $\overset{h}{\omega}_i \wedge d|\xi|^2 = 0$, since $\Omega_\rho G$ is defined by the equation $|\xi|^2 = \rho^2 = \text{const}$. From (2.7.4), we obtain

$$\overset{h}{\omega}_i \wedge d|\xi|^2 = \overset{h}{\omega}_i \wedge d(g_{kl}\xi^k\xi^l) = g\left[(-1)^{n+i-1}d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n + \right.$$

$$\left. + \sum_{j=1}^{n}(-1)^j\Gamma_{ip}^j\,\xi^p d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx\right] \wedge \left(\frac{\partial g_{kl}}{\partial x^r}\xi^k\xi^l dx^r + 2g_{kl}\xi^k d\xi^l\right) =$$

$$= g\left[(-1)^{n+i-1}\frac{\partial g_{kl}}{\partial x^r}\xi^k\xi^l d\xi \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \wedge dx^r + \right.$$

$$\left. +2\sum_{j=1}^{n}(-1)^j g_{kl}\Gamma_{ip}^j\,\xi^p\xi^k d\xi^1 \wedge \ldots \wedge \widehat{d\xi^j} \wedge \ldots \wedge d\xi^n \wedge dx \wedge d\xi^l\right] =$$

$$= \left(-\frac{\partial g_{kl}}{\partial x^i}\xi^k\xi^l + 2g_{kl}\Gamma_{ip}^l\xi^p\xi^k\right)dV^{2n}.$$

After an evident transformation, the obtained result can be rewritten as:

$$\overset{h}{\omega}_i \wedge d|\xi|^2 = \left(g_{ip}\Gamma_{kl}^p + g_{kp}\Gamma_{il}^p - \frac{\partial g_{kl}}{\partial x^i}\right)\xi^k\xi^l dV^{2n}.$$

The expression in parentheses on the right-hand side of this equality is equal to zero, as follows from (2.3.4).

Thus, for $D = T_{1,\rho}G$, formula (2.7.11) assumes the form

$$\int_{T_{1,\rho}G} \overset{h}{\nabla}_i u^i\, dV^{2n} = \int_{T_{1,\rho}(\partial G)} u^i\overset{h}{\omega}_i. \tag{2.7.14}$$

Let ∂G be smooth near a point $x_0 \in \partial G$. We can choose a coordinate system $(x^1, \ldots, x^n)$ in a neighborhood of the point x_0 in such a way that $g_{in} = \delta_{in}$, ∂G is determined by the equation $x^n = 0$ and $x^n > 0$ outside G (it is one of the so-called *semigeodesic* coordinate systems of the hypersurface ∂G). In these coordinates $\overset{h}{\omega}_\alpha = 0$ ($1 \le \alpha \le n-1$) on $T_{1,\rho}(\partial G)$, as follows from (2.7.4). One can easily see that the form

$$dV^{2n-1} = \overset{h}{\omega}_n = -g\, d\xi \wedge dx^1 \wedge \ldots \wedge dx^{n-1}, \tag{2.7.15}$$

is independent of the arbitrariness in the choice of the indicated coordinate system and, consequently, is defined globally on $T_{1,\rho}(\partial G)$. It is natural to call this form the *volume form of the manifold* $T_{1,\rho}(\partial G)$, since $dx^n \wedge dV^{2n-1} = dV^{2n}$. Written in the above coordinate system, the integrand of the right-hand side of equality (2.7.14) takes the form $u^i\overset{h}{\omega}_i = u^n\overset{h}{\omega}_n = \langle u, \nu\rangle\, dV^{2n-1}$, where ν is the unit vector of the outer normal to the boundary. Thus formula (2.7.14) can be written as:

$$\int_{T_{1,\rho}G} \overset{h}{\nabla}_i u^i\, dV^{2n} = \int_{T_{1,\rho}(\partial G)} \langle u, \nu\rangle\, dV^{2n-1}. \tag{2.7.16}$$

We will carry out further simplification of formula (2.7.16) under the assumption that the semibasic vector field $u = u(x,\xi)$ is positively homogeneous in its second argument

$$u(x, t\xi) = t^\lambda u(x,\xi) \qquad (t > 0). \tag{2.7.17}$$

In this case the integrands on (2.7.16) are homogeneous in ξ, and we will make use of this fact. To this end, we consider the $2n - 1$-form

$$d\Sigma^{2n-1} = \frac{g}{|\xi|}\sum_{i=1}^{n}(-1)^i\xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n \wedge dx \tag{2.7.18}$$

which is defined on TM for $\xi \ne 0$. It is natural to call its restriction to $\Omega_\rho M$ the *volume form of the manifold* $\Omega_\rho M$, since $d|\xi| \wedge d\Sigma^{2n-1} = dV^{2n}$.

First we transform the left-hand side of formula (2.7.16). To this end we define the diffeomorphism

$$\chi : [1, \rho] \times \Omega G \to T_{1,\rho}G, \qquad \chi(t; x, \xi) = (x, t\xi). \tag{2.7.19}$$

It satisfies the equality

$$\left[\chi^*(\overset{h}{\nabla}_i u^i\, dV^{2n})\right](t;x,\xi) = t^{\lambda+n-1}(\overset{h}{\nabla}_i u^i)(x,\xi)\, dt \wedge d\Sigma^{2n-1}(x,\xi). \tag{2.7.20}$$

(Henceforth in this section, given a smooth mapping $f : X \to Y$ and a differential form α on Y, by $f^*\alpha$ we mean the pull-back of α.) Indeed, (2.7.19) implies that $\chi^*(dx^i) = dx^i$, $\chi^*(d\xi^i) = t\, d\xi^i + \xi^i dt$. Thus,

$$\chi^*(\overset{h}{\nabla}_i u^i\, dV^{2n}) = \chi^*[g\overset{h}{\nabla}_i u^i\, d\xi \wedge dx] = g(\overset{h}{\nabla}_i u^i)\,(x,t\xi)\,(t\,d\xi^1 + \xi^1 dt) \wedge \ldots \wedge (t\,d\xi^n + \xi^n dt) \wedge dx.$$

By (2.7.17), $(\overset{h}{\nabla}_i u^i)(x,t\xi) = t^\lambda(\overset{h}{\nabla}_i u^i)(x,\xi)$, and the previous formula takes the form

$$\chi^*(\overset{h}{\nabla}_i u^i\, dV^{2n}) = t^\lambda g(\overset{h}{\nabla}_i u^i)(x,\xi)\left[t^n d\xi + t^{n-1}dt \wedge \sum_{i=1}^{n-1}(-1)^{i-1}\xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n\right] \wedge dx. \tag{2.7.21}$$

By the equality $|\xi|^2 = 1$, the relation $\xi_i d\xi^i = 0$ is valid on ΩG, and, consequently, $d\xi = 0$. Taking into account the last equality and (2.7.18), we see that (2.7.21) is equivalent to (2.7.20).

With the help of (2.7.20), the left-hand side of formula (2.7.16) is transformed as follows:

$$\int\limits_{T_{1,\rho}G} \overset{h}{\nabla}_i u^i\, dV^{2n} = \int\limits_{[1,\rho]\times\Omega G} \chi^*\left(\overset{h}{\nabla}_i u^i\, dV^{2n}\right) = \int\limits_1^\rho t^{\lambda+n-1}dt \int\limits_{\Omega G} \overset{h}{\nabla}_i u^i\, d\Sigma^{2n-1} = \frac{\rho^{\lambda+n}-1}{\lambda+n}\int\limits_{\Omega G}\overset{h}{\nabla}_i u^i\, d\Sigma^{2n-1}. \tag{2.7.22}$$

To fulfil a similar transformation of the right-hand side of equality (2.7.18) we introduce the manifold $\partial\Omega G = \{(x,\xi) \in TM \mid x \in \partial G,\ |\xi| = 1\}$ and consider the diffeomorphism

$$\chi : [1,\rho] \times \partial\Omega G \to T_{1,\rho}(\partial G); \qquad \chi(t;x,\xi) = (x,t\xi),$$

which is the restriction of the diffeomorphism (2.7.19) to $[1,\rho]\times\partial\Omega G$. Let $(x^1,\ldots,x^n)$ be the semigeodesic coordinate system used in definition (2.7.15) of the form dV^{2n-1}. In full analogy with the proof of equality (2.7.20), the next relation is verified:

$$[\chi^*(\langle u,\nu\rangle dV^{2n-1})](t;x,\xi) = t^{\lambda+n-1}\langle u,\nu\rangle(x,\xi)\, dt \wedge d\Sigma^{2n-2}(x,\xi), \tag{2.7.23}$$

where the form $d\Sigma^{2n-2}$ is defined in the indicated coordinate system by the equality

$$d\Sigma^{2n-2} = g\sum_{i=1}^n(-1)^i\xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n \wedge dx^1 \wedge \ldots \wedge dx^{n-1}. \tag{2.7.24}$$

One can easily see that this form is independent of the arbitrariness in the choice of our coordinate system and, consequently, is defined globally on $\partial\Omega G$. It is natural to call this form the *volume form of the manifold* $\partial\Omega G$, since $d|\xi| \wedge d\Sigma^{2n-2} = dV^{2n-1}$, as follows from (2.7.15) and (2.7.24). With the help of (2.7.24), the right-hand side of (2.7.18) takes the form:

$$\int\limits_{T_{1,\rho}(\partial G)} \langle u,\nu\rangle\, dV^{2n-1} = \frac{\rho^{\lambda+n}-1}{\lambda+n}\int\limits_{\partial\Omega G} \langle u,\nu\rangle\, d\Sigma^{2n-2}. \tag{2.7.25}$$

Inserting (2.7.22) and (2.7.25) into (2.7.16), we arrive at the final version of the Gauss-Ostrogradskiĭ formula for the horizontal divergence:

$$\int\limits_{\Omega G} \overset{h}{\nabla}_i u^i\, d\Sigma^{2n-1} = \int\limits_{\partial\Omega G} \langle u,\nu\rangle\, d\Sigma^{2n-2}. \tag{2.7.26}$$

The above-presented proof of formula (2.7.26) was fulfilled under the assumption that $\lambda + n \neq 0$. Nevertheless, the formula is valid for an arbitrary λ. Indeed, for $\lambda + n = 0$, the factor $(\rho^{\lambda+n}-1)/(\lambda+n)$ in equalities (2.7.22) and (2.7.25) is replaced by $\ln\rho - 1$; the remainder of the proof goes through without change.

The forms $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ participating in relation (2.7.26) have a simple geometrical sense. To clarify it we note that, for every point $x \in M$, the tangent space $T_x M$ is provided by the structure of a

Euclidean vector space which is induced by the Riemannian metric. By $dV_x^n(\xi)$ we denote the Euclidean volume form on $T_x M$. In a local coordinate system it is expressed by the formula

$$dV_x^n(\xi) = g^{1/2} d\xi^1 \wedge \ldots \wedge d\xi^n = g^{1/2} d\xi. \tag{2.7.27}$$

By $d\omega_x(\xi)$ we denote the angle measure, on the unit sphere $\Omega_x M = \{\xi \in T_x M \mid |\xi| = 1\}$ of the space $T_x M$, induced by the Euclidean structure of the space. In coordinates this form is expressed as follows:

$$d\omega_x(\xi) = g^{1/2} \sum_{i=1}^{n} (-1)^{i-1} \xi^i d\xi^1 \wedge \ldots \wedge \widehat{d\xi^i} \wedge \ldots \wedge d\xi^n. \tag{2.7.28}$$

This equality can be verified with the help of Lemma 2.7.3. Indeed, it follows from (2.7.27) and (2.7.28) that, for $|\xi| = 1$, the relation $d|\xi| \wedge d\omega_x(\xi) = dV_x^n(\xi)$ is valid. The last equality is just the definition of the angle measure on $\Omega_x M$.

Comparing definitions (2.7.18) and (2.7.24) of the forms $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ with equality (2.7.28), we see that

$$d\Sigma^{2n-1} = d\omega_x(\xi) \wedge dV^n(x), \quad d\Sigma^{2n-2} = (-1)^n d\omega_x(\xi) \wedge dV^{n-1}(x) \tag{2.7.29}$$

where $dV^n(x) = g^{1/2} dx$ is the Riemannian volume form on M and $dV^{n-1}(x)$ is the Riemannian volume form on ∂G. In the semigeodesic coordinate system have been used in definition (2.7.24), the last form is given by the formula $dV^{n-1}(x) = g^{1/2} dx^1 \wedge \ldots \wedge dx^{n-1}$.

Substituting the expressions (2.7.29) for $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ into (2.7.26), we obtain (2.7.2).

Lecture 3
The ray transform

In the first section we pose the problem of determining a simple Riemannian metric on a compact manifold with boundary from known distances in this metric between boundary points. This geometrical problem is interesting from the theoretical and applied points of view. Here it is considered as an example of a question leading to an integral geometry problem for a tensor field. In fact, by linearization of the problem we arrive at the question of finding a symmetric tensor field of degree 2 from its integrals over all geodesics of a given Riemannian metric. The operator sending a tensor field into the family of its integrals over all geodesics is called the ray transform. The principal difference between scalar and tensor integral geometry is that in the last case the operators under consideration have, as a rule, nontrivial kernels. It is essential that in the process of linearization there arises a conjecture on the kernel of the ray transform.

In Section 3.2 we introduce a class of so-called dissipative Riemannian metrics. The ray transform can be defined in a natural way for dissipative metrics. This class essentially extends the class of simple metrics.

In Section 3.3 we define the ray transform on a compact dissipative Riemannian manifold and prove that it is bounded with respect to the Sobolev norms.

Integral geometry is well known to be closely related to inverse problems for kinetic and transport equations. In Section 3.4 we introduce the kinetic equation on a Riemannian manifold and show that the integral geometry problem for a tensor field is equivalent to an inverse problem of determining the source, in the kinetic equation, which depends polynomially on a direction.

Section 3.5 contains a survey of some results that are related to the questions under consideration but are not mentioned in the main part of our lectures.

3.1   The boundary rigidity problem

The general boundary rigidity problem reads: to which extent is a Riemannian metric on a compact manifold with boundary determined from the distances between boundary points? More precisely, it can be formulated as follows.

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let g' be another Riemannian metric on M. We say that g and g' have the same *boundary distance-function* if $\Gamma_g(x, y) = \Gamma_{g'}(x, y)$ for arbitrary boundary points $x, y \in \partial M$, where Γ_g (resp. $\Gamma_{g'}$) represents distance in M with respect to g (resp. g'). It is easy to give examples of pairs of metrics with the same boundary distance-function. Indeed, if $\varphi : M \to M$ is an arbitrary diffeomorphism of M onto itself which is the identity on the boundary, then the metrics g and $g' = \varphi^* g$ have the same boundary distance-function. Here $g' = \varphi^* g$ is the pull-back of g under φ; (i.e., for arbitrary vectors $\xi, \eta \in T_x M$ we have $\langle \xi, \eta \rangle'_x = \langle \varphi_* \xi, \varphi_* \eta \rangle_{\varphi(x)}$, where $\varphi_* : T_x M \to T_{\varphi(x)} M$ is the differential of φ at x and $\langle \, , \, \rangle$ (resp. $\langle \, , \, \rangle'$) is the inner product with respect to the metric g (resp. g')).

We say that a compact Riemannian manifold is *boundary rigid* if this is the only type of nonuniqueness. More precisely, (M, g) is *boundary rigid* if for every Riemannian metric g' on M with the same boundary distance-function as g, there is a diffeomorphism $\varphi : M \to M$ which is the identity on the boundary and for which $g' = \varphi^* g$.

The next question of stability in this problem seems to be important as well: are two metrics close (in some sense) to each other in the case when their boundary distance functions are close?

There are evident examples of manifolds that are not boundary rigid. For instance, let $M = \{(x \in \mathbf{R}^3 \mid x_1^2 + x_2^2 = 2x_3, \ x_3 \le a\}$ be the part of the paraboloid with the metric induced from $\mathbf{R}^3$. For a sufficiently large a, every minimizing geodesic joining boundary points $x, y \in \partial M = \{x \in M \mid x_3 = a\}$

does not intersect a neighborhood of the vertex $(0,0,0)$. Therefore we can change the metric in the neighborhood without changing the boundary distance function.

So, the boundary rigidity problem should be considered under some additional assumptions on geometry of geodesics. We will restrict ourselves to considering simple metrics. Roughly speaking, simplicity of a metric means that geodesics constitute a regular family of curves in the sense of Section 1.1. Let us give precise definitions.

Let M be a Riemannian manifold with boundary ∂M. For a point $x \in \partial M$, the *second quadratic form of the boundary*

$$\mathrm{II}(\xi, \xi) = \langle \nabla_\xi \nu, \xi \rangle \qquad (\xi \in T_x(\partial M))$$

is defined on the space $T_x(\partial M)$ where $\nu = \nu(x)$ is the unit outer normal vector to the boundary. We say that the boundary is *strictly convex* if the form is positive-definite for all $x \in \partial M$.

A compact Riemannian manifold (M, g) with boundary (or the metric g) is called *simple* if (1) the boundary is strictly convex, and (2) every two points $x, y \in M$ are joint by a unique geodesic smoothly depending on x and y. The latter means that the mapping $\exp_x^{-1} : M \to T_x M$ is smooth.

Problem 3.1.1 (the boundary rigidity problem) *Is any simple Riemannian manifold (M, g) boundary rigid? In other words, does the equality $\Gamma_g = \Gamma_{g'}$, for another simple metric g' on M, imply existence of a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\partial M} = \mathrm{Id}$ and $\varphi^* g = g'$?*

Until now a positive answer to this question is obtained for rather narrow classes of metrics (there is a survey of such results in Section 3.5; some new results in this direction are obtained in Lecture 5). On the other hand, I do not know any counterexample to this conjecture.

Let us linearize Problem 3.1.1. To this end we suppose g^τ to be a family, of simple metrics on M, smoothly depending on $\tau \in (-\varepsilon, \varepsilon)$. Let us fix $p, q \in \partial M$, $p \neq q$, and put $a = \Gamma_{g^0}(p, q)$. Let $\gamma^\tau : [0, a] \to M$ be the geodesic, of the metric g^τ, for which $\gamma^\tau(0) = p$ and $\gamma^\tau(a) = q$. Let $\gamma^\tau = (\gamma^1(t, \tau), \dots, \gamma^n(t, \tau))$ be the coordinate representation of γ^τ in a local coordinate system, $g^\tau = (g_{ij}^\tau)$. Simplicity of g^τ implies smoothness for the functions $\gamma^i(t, \tau)$. The equality

$$\frac{1}{a}[\Gamma_{g^\tau}(p, q)]^2 = \int_0^a g_{ij}^\tau(\gamma^\tau(t)) \dot\gamma^i(t, \tau) \dot\gamma^j(t, \tau) dt \qquad (3.1.1)$$

is valid in which the dot denotes differentiation with respect to t. Differentiating (3.1.1) with respect to τ and putting then $\tau = 0$, we get

$$\frac{1}{a}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}[\Gamma_{g^\tau}(p, q)]^2 = \int_0^a f_{ij}(\gamma^0(t))\dot\gamma^i(t, 0)\dot\gamma^j(t, 0)\, dt + \int_0^a \frac{\partial}{\partial\tau}\bigg|_{\tau=0} \left[g_{ij}^0(\gamma^\tau(t))\dot\gamma^i(t, \tau)\dot\gamma^j(t, \tau) \right] dt \quad (3.1.2)$$

where

$$f_{ij} = \frac{\partial}{\partial\tau}\bigg|_{\tau=0} g_{ij}^\tau. \qquad (3.1.3)$$

The second integral on the right-hand side of (3.1.2) is equal to zero since the geodesic γ^0 is an extremal of the functional

$$E_0(\gamma) = \int_0^a g_{ij}^0(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\, dt.$$

Thus we come to the equality

$$\frac{1}{a}\frac{\partial}{\partial\tau}\bigg|_{\tau=0}[\Gamma_{g^\tau}(p, q)]^2 = If(\gamma_{pq}) \equiv \int_{\gamma_{pq}} f_{ij}(x)\dot x^i \dot x^j\, dt \qquad (3.1.4)$$

in which γ_{pq} is a geodesic of the metric g^0 and t is the arc length of this geodesic in the metric g^0.

If the boundary distance function Γ_{g^τ} does not depend on τ, then the left-hand side of (3.1.4) is equal to zero. On the other hand, if Problem 3.1.1 has a positive answer for the family g^τ, then there exists a one-parameter family of diffeomorphisms $\varphi^\tau : M \to M$ such that $\varphi^\tau|_{\partial M} = \mathrm{Id}$ and $g^\tau = (\varphi^\tau)^* g^0$. Written in coordinate form, the last equality gives

$$g_{ij}^\tau = (g_{kl}^0 \circ \varphi^\tau)\frac{\partial\varphi^k(x, \tau)}{\partial x^i}\frac{\partial\varphi^l(x, \tau)}{\partial x^j}$$

where $\varphi^\tau(x) = (\varphi^1(x,\tau), \ldots, \varphi^n(x,\tau))$. Differentiating this relation with respect to τ and putting $\tau = 0$, we get the equation

$$(dv)_{ij} \equiv \frac{1}{2}(v_{i\,;\,j} + v_{j\,;\,i}) = \frac{1}{2}f_{ij}, \tag{3.1.5}$$

for the vector field $\frac{d}{d\tau}|_{\tau=0}\varphi^\tau$ where $v_{i\,;\,j}$ are covariant derivatives of the field v in the metric g^0. The condition $\varphi^\tau|_{\partial M} = \text{Id}$ implies that $v|_{\partial M} = 0$. Thus we come to the following question which is a linearization of Problem 3.1.1: to what extent is a symmetric tensor field $f = (f_{ij})$ on a simple Riemannian manifold (M, g^0) determined by the family of integrals (3.1.4) which are known for all $p, q \in \partial M$? In particular, is it true that the equality $If(\gamma_{pq}) = 0$ for all $p, q \in \partial M$ implies existence of a vector field v such that $v|_{\partial M} = 0$ and $dv = f$? In the latter case (M, g^0) is called a *deformation boundary rigid* manifold.

Let us generalize this linear problem to tensor fields of arbitrary degree. To this end we note that the operator d defined by (3.1.5) is the special case of operator (2.4.1) that represents the symmetric part of the covariant derivative of a symmetric tensor field. In full analogy with the above considerations, we pose the following

Problem 3.1.2 (the integral geometry problem for tensor fields) *Let (M, g) be a simple Riemannian manifold, and $m \geq 0$ be an integer. To what extent is a symmetric tensor field $f \in C^\infty(S^m \tau'_M)$ determined by the set of the integrals*

$$If(\gamma_{pq}) = \int\limits_{\gamma_{pq}} f_{i_1 \ldots i_m}(x)\dot{x}^{i_1} \ldots \dot{x}^{i_m}\, dt \tag{3.1.6}$$

that are known for all $p, q \in \partial M$? Here γ_{pq} is the geodesic with endpoints p, q and t is the arc length on this geodesic. In particular, does the equality $If(\gamma_{pq}) = 0$ for all $p, q \in \partial M$ imply existence of a field $v \in C^\infty(S^{m-1}\tau'_M)$, such that $v|_{\partial M} = 0$ and $dv = f$?

By the *ray transform* of the field f we will mean the function If that is determined by formula (3.1.6) on the set of geodesics joining boundary points. In Section 3.3 this problem will be generalized to a wider class of metrics. Note that in the case of $m = 0$ this is the integral geometry problem for a scalar function and the regular family of geodesics, just the problem considered in Section 1.1 in the two-dimensional case.

3.2 Compact dissipative Riemannian manifolds

A compact Riemannian manifold M with boundary is called a *compact dissipative Riemannian manifold* (CDRM briefly), if it satisfies two conditions: 1) the boundary ∂M is strictly convex; 2) for every point $x \in M$ and every vector $0 \neq \xi \in T_x M$, the maximal geodesic $\gamma_{x,\xi}(t)$ satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \xi$ is defined on a finite segment $[\tau_-(x,\xi), \tau_+(x,\xi)]$. We recall simultaneously that a geodesic $\gamma : [a,b] \to M$ is called maximal if it cannot be extended to a segment $[a - \varepsilon_1, b + \varepsilon_2]$, where $\varepsilon_i \geq 0$ and $\varepsilon_1 + \varepsilon_2 > 0$.

The second of the conditions participating in the definition of CDRM is equivalent to the absence of a geodesic of infinite length in M.

Recall that by $TM = \{(x,\xi) \mid x \in M,\ \xi \in T_x M\}$ we denote the space of the tangent bundle of the manifold M, and by $\Omega M = \{(x,\xi) \in TM \mid |\xi| = 1\}$ we denote its submanifold that consists of unit vectors. We introduce the next submanifolds of TM:

$$T^0 M = \{(x,\xi) \in TM \mid \xi \neq 0\},$$

$$\partial_\pm \Omega M = \{(x,\xi) \in \Omega M \mid x \in \partial M;\ \pm\langle \xi, \nu(x) \rangle \geq 0\},$$

where ν is the unit vector of the outer normal to the boundary. Note that $\partial_+ \Omega M$ and $\partial_- \Omega M$ are compact manifolds with the common boundary $\partial_0 \Omega M = \Omega M \bigcap T(\partial M)$, and $\partial \Omega M = \partial_+ \Omega M \bigcup \partial_- \Omega M$.

While defining a CDRM, we have determined two functions $\tau_\pm : T^0 M \to \mathbf{R}$. It is evident that they have the next properties:

$$\gamma_{x,\xi}(\tau_\pm(x,\xi)) \in \partial M; \tag{3.2.1}$$

$$\tau_+(x,\xi) \geq 0,\quad \tau_-(x,\xi) \leq 0,\quad \tau_+(x,\xi) = -\tau_-(x,-\xi);$$

$$\tau_\pm(x, t\xi) = t^{-1}\tau_\pm(x,\xi)\quad (t > 0); \tag{3.2.2}$$

$$\tau_+|_{\partial_+\Omega M} = \tau_-|_{\partial_-\Omega M} = 0.$$

We now consider the smoothness properties of the functions $\tau_\pm$. With the help of the implicit function theorem, one can easily see that $\tau_\pm(x,\xi)$ is smooth near a point (x,ξ) such that the geodesic $\gamma_{x,\xi}(t)$ intersects ∂M transversely for $t = \tau_\pm(x,\xi)$. By strict convexity of ∂M, the last claim is valid for all $(x,\xi) \in T^0M$ except for the points of the set $\partial_0 T^0 M = T^0 M \bigcap T(\partial M)$. Thus we conclude that $\tau_\pm$ are smooth on $T^0 M \setminus \partial_0 T^0 M$. All points of the set $\partial_0 T^0 M$ are singular points for $\tau_\pm$, since one can easily see that some derivatives of these functions are unbounded in a neighborhood of such a point. Nevertheless, the next claim is valid:

Lemma 3.2.1 *Let (M,g) be a CDRM. The function $\tau : \partial \Omega M \to \mathbf{R}$ defined by the equality*

$$\tau(x,\xi) = \begin{cases} \tau_+(x,\xi), & \text{if} \quad (x,\xi) \in \partial_- \Omega M, \\ \tau_-(x,\xi), & \text{if} \quad (x,\xi) \in \partial_+ \Omega M \end{cases} \tag{3.2.3}$$

is smooth. In particular, $\tau_- : \partial_+ \Omega M \to \mathbf{R}$ is a smooth function.

Proof. In some neighborhood U of a point $x_0 \in \partial M$, a semigeodesic coordinate system $(x^1, \ldots, x^n) = (y^1, \ldots, y^{n-1}, r)$ can be introduced such that the function r coincides with the distance (in the metric g) from the point (y,r) to ∂M and $g_{in} = \delta_{in}$. In this coordinate system, the Christoffel symbols satisfy the relations $\Gamma_{in}^n = \Gamma_{nn}^i = 0$, $\Gamma_{\beta n}^\alpha = -g^{\alpha\gamma}\Gamma_{\beta\gamma}^n$ (in this and subsequent formulas, Greek indices vary from 1 to $n-1$; on repeating Greek indices, the summation from 1 to $n-1$ is assumed), the unit vector of the outer normal has the coordinates $(0,\ldots,0,-1)$. Putting $j = n$ in (2.2.6), we see that the Christoffel symbols $\Gamma_{\alpha\beta}^n$ coincide with the coefficients of the second quadratic form. Consequently, the condition of strict convexity of the boundary means that

$$\Gamma_{\alpha\beta}^n(y,0)\eta^\alpha\eta^\beta \geq a|\eta|^2 = a\sum_{\alpha=1}^{n-1}(\eta^\alpha)^2 \qquad (a > 0). \tag{3.2.4}$$

Let $(y^1, \ldots, y^{n-1}, r, \eta^1, \ldots, \eta^{n-1}, \rho)$ be the coordinate system on TM associated with $(y^1, \ldots, y^{n-1}, r)$. As we have seen before the formulation of the lemma, the function $\tau(y,0,\eta,\rho)$ is smooth for $\rho \neq 0$. Consequently, to prove the lemma it is sufficient to verify that this function is smooth for $|\eta| \geq 1/2$ and $|\rho| < \varepsilon$ with some $\varepsilon > 0$.

Let $\gamma_{(y,\eta,\rho)}(t) = (\gamma_{(y,\eta,\rho)}^1(t), \ldots, \gamma_{(y,\eta,\rho)}^n(t))$ be the geodesic defined by the initial conditions $\gamma_{(y,\eta,\rho)}(0) = (y,0)$, $\dot\gamma_{(y,\eta,\rho)}(0) = (\eta,\rho)$. Expanding the function $r(t,y,\eta,\rho) = \gamma_{(y,\eta,\rho)}^n(t)$ into the Taylor series in t and using equations (2.3.5) for geodesics, we obtain the representation

$$r(t,y,\eta,\rho) = \rho t - \frac{1}{2}\Gamma_{\alpha\beta}^n(y,0)\eta^\alpha\eta^\beta t^2 + \varphi(t,y,\eta,\rho)t^3 \tag{3.2.5}$$

with some smooth function $\varphi(t,y,\eta,\rho)$. For small ρ, the equation $r(t,y,\eta,\rho) = 0$ has the solutions $t = 0$ and $t = \tau(y,0,\eta,\rho)$. Consequently, (3.2.5) implies that $\tau = \tau(y,0,\eta,\rho)$ is a solution to the equation

$$F(\tau,y,\eta,\rho) \equiv \rho - \frac{1}{2}\Gamma_{\alpha\beta}^n(y,0)\eta^\alpha\eta^\beta\tau + \varphi(\tau,y,\eta,\rho)\tau^2 = 0.$$

It follows from (3.2.4) that $\frac{\partial}{\partial\tau}\Big|_{\tau=0} F(\tau,y,\eta,\rho) \neq 0$. Applying the implicit function theorem, we see that $\tau(y,0,\eta,\rho)$ is a smooth function. The lemma is proved.

Lemma 3.2.2 *Let M be a CDRM. The function $\tau_+(x,\xi)/(-\langle\xi,\nu(x)\rangle)$ is bounded on the set $\partial_-\Omega M \setminus \partial_0\Omega M$.*

Proof. It suffices to prove that the function is bounded on the subset $W_\varepsilon = \{(x,\xi) \mid 0 < -\langle\xi,\nu(x)\rangle < \varepsilon, \ 1/2 \leq |\xi| \leq 3/2\}$ of the manifold $\partial(TM)$ for some $\varepsilon > 0$. Decreasing ε, one can easily see that it suffices to verify boundedness of the function for $(x,\xi) \in W_\varepsilon$ such that the geodesic $\gamma_{x,\xi} : [0, \tau_+(x,\xi)] \to M$ is wholly in the domain U of the semigeodesic coordinate system introduced in the proof of Lemma 3.2.1. In these coordinates, $(x,\xi) = (y,0,\eta,\rho)$, $0 < -\langle\xi,\nu(x)\rangle = \rho < \varepsilon$, $1/2 \leq |\eta| \leq 3/2$. The left-hand side of equality (3.2.5) vanishes for $t = \tau_+(x,\xi)$:

$$\left[\frac{1}{2}\Gamma_{\alpha\beta}^n(y,0)\eta^\alpha\eta^\beta - \varphi(\tau_+(x,\xi),y,\eta,\rho)\tau_+(x,\xi)\right]\frac{\tau_+(x,\xi)}{\rho} = 1. \tag{3.2.6}$$

By decreasing ε, we can achieve that $\tau_+(x,\xi) < \delta$ for $(x,\xi) \in W_\varepsilon$ with any $\delta > 0$. Thus the second summand in the brackets of (3.2.6) can be made arbitrarily small. Together with (3.2.4), this implies that the expression in the brackets is bounded from below by some positive constant. Consequently, $0 < -\tau_+(x,\xi)/\langle \xi, \nu(x)\rangle = \tau_+(x,\xi)/\rho \leq C$. The lemma is proved.

We will need the next claim in Section 4.3.

Lemma 3.2.3 *Let (M,g) be a CDRM and $x_0 \in \partial M$. Let a semigeodesic coordinate system $(x^1, \ldots, x^n)$ be chosen in a neighborhood U of the point x_0 in such a way that x^n coincides with the distance in the metric g from x to ∂M, and let $(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ be the associated coordinate system on TM. There exists a neighborhood $U' \subset U$ of the point x_0 such that the derivatives*

$$\frac{\partial \tau_-(x,\xi)}{\partial x^\alpha} \quad (\alpha = 1, \ldots, n-1); \qquad \frac{\partial \tau_-(x,\xi)}{\partial \xi^i} \quad (i = 1, \ldots, n) \tag{3.2.7}$$

are bounded on the set $\Omega M \bigcap p^{-1}(U' \setminus \partial M)$, where $p : TM \to M$ is the projection of the tangent bundle.

Proof. It suffices to prove boundedness of derivatives (3.2.7) only for $(x,\xi) \in \Omega M \bigcap p^{-1}(U' \setminus \partial M)$ such that the geodesic $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is wholly in U. By $\gamma^i(t, x, \xi)$ we denote the coordinates of the point $\gamma_{x,\xi}(t)$. The point $\gamma_{x,\xi}(\tau_-(x,\xi))$ is in ∂M. This means that $\gamma^n(\tau_-(x,\xi), x, \xi) = 0$. Differentiating the last equality, we obtain

$$\frac{\partial \tau_-(x,\xi)}{\partial x^i} = -\frac{\partial \gamma^n}{\partial x^i}(\tau_-, x, \xi)/\dot\gamma^n(\tau_-, x, \xi), \quad \frac{\partial \tau_-(x,\xi)}{\partial \xi^i} = -\frac{\partial \gamma^n}{\partial \xi^i}(\tau_-, x, \xi)/\dot\gamma^n(\tau_-, x, \xi). \tag{3.2.8}$$

Note that $(\partial \gamma^n/\partial x^\alpha)(0, x, \xi) = 0$ for $1 \leq \alpha \leq n-1$. Consequently, a representation $(\partial \gamma^n/\partial x^\alpha)(\tau_-, x, \xi) = \varphi_\alpha(\tau_-, x, \xi)\tau_-(x,\xi)$ is possible with some functions $\varphi_\alpha(t', x, \xi)$ smooth on the set

$$W = \{(t', x, \xi) \in \mathbf{R} \times T^0 M \mid \tau_-(x,\xi) \leq t' \leq 0, \ \gamma_{x,\xi}(t) \in U \text{ for } \tau_-(x,\xi) \leq t \leq 0\}.$$

By the equality $(\partial \gamma^n/\partial \xi^i)(0, x, \xi) = 0$ $(1 \leq i \leq n)$, a representation

$$(\partial \gamma^n/\partial \xi^i)(\tau_-, x, \xi) = \psi_i(\tau_-, x, \xi)\tau_-(x,\xi)$$

is possible with some functions $\psi_i(t', x, \xi)$ smooth on W. Consequently, (3.2.8) is rewritten as

$$\frac{\partial \tau_-(x,\xi)}{\partial x^\alpha} = \varphi_\alpha(\tau_-, x, \xi)\frac{-\tau_-(x,\xi)}{\dot\gamma^n(\tau_-, x, \xi)}; \qquad \frac{\partial \tau_-(x,\xi)}{\partial \xi^i} = \psi_i(\tau_-, x, \xi)\frac{-\tau_-(x,\xi)}{\dot\gamma^n(\tau_-, x, \xi)}. \tag{3.2.9}$$

Since the functions φ_α and ψ_i are smooth on W, they are bounded on any compact subset of W. Consequently, (3.2.9) implies that the proof will be finished if we verify boundedness of the ratio $-\tau_-(x,\xi)/\dot\gamma^n(\tau_-(x,\xi), x, \xi)$ on $\Omega M \bigcap p^{-1}(U \setminus \partial M)$.

We denote $y = y(x,\xi) = \gamma_{x,\xi}(\tau_-(x,\xi))$, $\eta = \eta(x,\xi) = \dot\gamma_{x,\xi}(\tau_-(x,\xi))$; then $(y, \eta) \in \partial_-\Omega M \setminus \partial_0 \Omega M$, $0 \leq -\tau_-(x,\xi) \leq \tau_+(y, \eta)$ and $\dot\gamma^n(\tau_-(x,\xi), x, \xi) = -\langle \eta, \nu(y)\rangle$. Consequently,

$$0 \leq \frac{-\tau_-(x,\xi)}{\dot\gamma^n(\tau_-(x,\xi), x, \xi)} \leq \frac{\tau_+(y, \eta)}{-\langle \eta, \nu(y)\rangle}.$$

The last ratio is bounded on $\partial_-\Omega M \setminus \partial_0\Omega M$ by Lemma 3.2.2. The lemma is proved.

3.3 The ray transform on a CDRM

In definition (3.1.6) of the ray transform on a simple manifold, we parameterized the set of maximal geodesics by endpoins. Dealing with a CDRM, it is more comfortable to parameterize the set of maximal geodesics by points of the manifold $\partial_+\Omega M$.

Let $C^\infty(\partial_+\Omega M)$ be the space of smooth functions on the manifold $\partial_+\Omega M$.

The *ray transform* on a CDRM M is the linear operator

$$I : C^\infty(S^m \tau'_M) \to C^\infty(\partial_+\Omega M) \tag{3.3.1}$$

defined by the equality

$$If(x,\xi) = \int_{\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(t)), \dot\gamma_{x,\xi}^m(t)\rangle \, dt = \int_{\tau_-(x,\xi)}^{0} f_{i_1 \ldots i_m}(\gamma_{x,\xi}(t))\dot\gamma_{x,\xi}^{i_1}(t) \ldots \dot\gamma_{x,\xi}^{i_m}(t) \, dt, \tag{3.3.2}$$

where $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \xi$. By Lemma 3.2.1, the right-hand side of equality (3.3.2) is a smooth function on $\partial_+\Omega M$.

Recall that the Hilbert space $H^k(S^m\tau'_M)$ was introduced in Section 2.4. In a similar way the Hilbert space $H^k(\partial_+\Omega M)$ of functions on $\partial_+\Omega M$ is defined.

Theorem 3.3.1 *The ray transform on a CDRM is extendible to the bounded operator*

$$I : H^k(S^m\tau'_M) \to H^k(\partial_+\Omega M) \tag{3.3.3}$$

for every integer $k \geq 0$.

To prove the theorem we need the next

Lemma 3.3.2 (the Santalo formula) *Let (M, g) be a CDRM. For every function $\varphi \in C(\Omega M)$ the equality*

$$\int\limits_{\Omega M} \varphi(x,\xi)\, d\Sigma^{2n-1}(x,\xi) = \int\limits_{\partial_+\Omega M} \langle \xi, \nu(x) \rangle\, d\Sigma^{2n-2}(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} \varphi(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))\, dt \tag{3.3.4}$$

holds, where $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ are the volume forms on the manifolds ΩM and $\partial\Omega M$ respectively defined by formulas (2.7.18) and (2.7.24).

Proof. By the Liuville theorem [15], the volume form $d\Sigma^{2n-1}$ is preserved by the geodesic flow G^t.

We consider the domain $D = \{(x,\xi;t) \mid \tau_-(x,\xi) \leq t \leq 0\}$ in the manifold $\partial_+\Omega M \times \mathbf{R}$ and define a smooth mapping $G : D \to \Omega M$ by putting $G(x,\xi;t) = G^t(x,\xi)$, where G^t is the geodesic flow. It maps the interior of D diffeomorphically onto $\Omega M \setminus T(\partial M)$. Consequently,

$$\int\limits_{\Omega M} \varphi\, d\Sigma^{2n-1} = \int\limits_D (\varphi \circ G) G^*(d\Sigma^{2n-1}). \tag{3.3.5}$$

Differentiating the relation $(h\circ G)(x,\xi;t) = h(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))$ with respect to t, we obtain $\partial(h\circ G)/\partial t = (Hh) \circ G$, where H is the vector field (2.6.1) on TM generating the geodesic flow. Valid for every $h \in C^\infty(\Omega M)$, the last equality means that the vector fields $\partial/\partial t$ and H are G-connected (we recall that, given a diffeomorphism $f : X \to Y$ of two manifolds, vector fields $u \in C^\infty(\tau_X)$ and $v \in C^\infty(\tau_Y)$ are called f-*connected* if the differential of f transforms u into v; compare [33]). Since the form $d\Sigma^{2n-1}$ is preserved by the geodesic flow, $G^*(d\Sigma^{2n-1})$ is preserved by the flow of the field $\partial/\partial t$. This implies, as is easily seen, that $G^*(d\Sigma^{2n-1}) = d\sigma \wedge dt$ for some form $d\sigma = a\, d\Sigma^{2n-2}$ on $\partial_+\Omega M$ with $a \in C^\infty(\partial_+\Omega M)$. Thus (3.3.5) can be rewritten as:

$$\int\limits_{\Omega M} \varphi\, d\Sigma^{2n-1} = \int\limits_{\partial_+\Omega M} a(x,\xi) d\Sigma^{2n-2}(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} \varphi(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)) dt, \tag{3.3.6}$$

and

$$(G^*(d\Sigma^{2n-1}))(x,\xi;t) = a(x,\xi) d\Sigma^{2n-2}(x,\xi) \wedge dt. \tag{3.3.7}$$

To finish the proof, we should show that $a(x,\xi) = \langle \xi, \nu(x) \rangle$. To this end it suffices to prove that the equality

$$(G^*(d\Sigma^{2n-1}))(x,\xi;0) = \langle \xi, \nu(x) \rangle d\Sigma^{2n-2}(x,\xi) \wedge dt \tag{3.3.8}$$

holds for $t = 0$.

It follows from definitions (2.7.18) and (2.7.24) of the forms $d\Sigma^{2n-1}$ and $d\Sigma^{2n-2}$ that $d\Sigma^{2n-1} = d\Sigma^{2n-2} \wedge dr$ on $\partial\Omega M$, where $r(x) = -\rho(x, \partial M)$ and ρ is the distance in the metric g. The function r is smooth in some neighborhood of ∂M, and $\nabla r(x) = \nu(x)$ for $x \in \partial M$.

The differential of the mapping G at a point $(x,\xi;0)$ is identical on $T_{(x,\xi)}(\partial_+\Omega M)$ and maps the vector $\partial/\partial t$ into H. Consequently,

$$(G^*(d\Sigma^{2n-1}))(x,\xi;0) = (G^*(d\Sigma^{2n-2} \wedge dr))(x,\xi;0) = Hr \cdot d\Sigma^{2n-2}(x,\xi) \wedge dt. \tag{3.3.9}$$

By (2.6.1), $Hr = \xi^i \dfrac{\partial r}{\partial x^i} = \langle \xi, \nu(x) \rangle$. Inserting this expression into (3.3.9), we obtain (3.3.8). The lemma is proved.

Corollary 3.3.3 *Let (M, g) be a CDRM, $d\Sigma$ and $d\sigma$ be smooth volume forms (differential forms of the most degree that do not vanish at every point) on ΩM and $\partial_+\Omega M$ respectively. Let D be the closed domain in $\partial_+\Omega M \times \mathbf{R}$ defined by the equality $D = \{(x, \xi; t) \mid \tau_-(x, \xi) \le t \le 0\}$, and the mapping $G : D \to \Omega M$ be defined by $G(x, \xi; t) = G^t(x, \xi)$, where G^t is the geodesic flow. Then the equality*

$$(G^* d\Sigma)(x, \xi; t) = a(x, \xi; t)\langle \xi, \nu(x)\rangle d\sigma(x, \xi) \wedge dt \qquad (3.3.10)$$

holds on D with some function $a \in C^\infty(D)$ not vanishing at every point.

Indeed, only the coefficient a changes in (3.3.10) under the change of the volume form $d\Sigma$ or $d\sigma$. Therefore it suffices to prove the claim for the forms $d\Sigma = d\Sigma^{2n-1}$ and $d\sigma = d\Sigma^{2n-2}$. In this case (3.3.10) coincides with (3.3.8).

Proof of Theorem 3.3.1. Let us agree to denote various constants independent of f by the same letter C.

First we will prove the estimate

$$\|If\|_k \le C\|f\|_k \qquad (3.3.11)$$

for $f \in C^\infty(S^m\tau_M')$. To this end, we define the function $F \in C^\infty(\Omega M)$ by putting

$$F(x, \xi) = f_{i_1\dots i_m}(x)\xi^{i_1}\dots\xi^{i_m}.$$

The inequality

$$\|F\|_k \le C\|f\|_k \qquad (3.3.12)$$

is evident. With the help of F, equality (3.3.2) is rewritten as:

$$If(x, \xi) = IF(x, \xi) \equiv \int\limits_{\tau_-(x,\xi)}^{0} F(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))\, dt. \qquad (3.3.13)$$

By (3.3.12), to prove (3.3.4) it suffices to establish the estimate

$$\|IF\|_k \le C\|F\|_k. \qquad (3.3.14)$$

Since operator (3.3.13) is linear, it suffices to prove (3.3.14) for a function $F \in C^\infty(\Omega M)$ such that its support is contained in a domain $V \subset \Omega M$ of some local coordinate system $(z^1, \dots, z^{2n-1})$ on the manifold ΩM.

Let $(y^1, \dots, y^{2n-2})$ be a local coordinate system on $\partial_+\Omega M$ defined in a domain $U \subset \partial_+\Omega M$, and φ be a smooth function whose support is contained in U. To prove (3.3.14) it suffices to establish the estimate

$$\|\varphi \cdot IF\|_{H^k(U)} \le C\|F\|_{H^k(V)}. \qquad (3.3.15)$$

Differentiating (3.3.13), we obtain

$$D_y^\alpha[\varphi(x, \xi)IF(x, \xi)] = \sum_{\beta+\gamma=\alpha}(D_y^\gamma\varphi)(x, \xi)\int\limits_{\tau_-(x,\xi)}^{0}D_y^\beta[F(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))]\, dt + $$

$$+ \sum_{\substack{\beta+\gamma+\delta=\alpha \\ \delta<\alpha}} C_{\beta\gamma\delta}^\alpha \cdot (D_y^\beta\varphi)(x, \xi) \cdot (D_y^\gamma\tau_-)(x, \xi) \cdot D_y^\delta[F(\gamma_{x,\xi}(\tau_-(x, \xi)), \dot{\gamma}_{x,\xi}(\tau_-(x, \xi)))]. \qquad (3.3.16)$$

We will prove that, for $|\alpha| \le k$, the L_2-norm of each of the summands on the right-hand side of (3.3.16) can be estimated by $C\|F\|_{H^k(V)}$.

By Lemma 3.2.1, the functions $D_y^\gamma\tau_-$ are locally bounded, and the mapping

$$\partial_+\Omega M \to \partial_-\Omega M, \qquad (x, \xi) \mapsto (\gamma_{x,\xi}(\tau_-(x, \xi)), \dot{\gamma}_{x,\xi}(\tau_-(x, \xi)))$$

is a diffeomorphism. Therefore the L_2-norm of the second sum on the right-hand side of (3.3.16) is not more than $C\|F|_{\partial_-\Omega M}\|_{k-1}$. Using the boundedness of the trace operator $H^k(\Omega M) \to H^{k-1}(\partial_-\Omega M)$, $F \mapsto F|_{\partial_-\Omega M}$, we conclude that the L_2-norm of the second sum on the right-hand side of (3.3.16) is majorized by $C\|F\|_{H^k(V)}$.

We now estimate the L_2-norm of the integral on the right-hand side of (3.3.16). With the help of the Cauchy-Bunyakovskiĭ inequality, we obtain

$$\left| \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))] \, dt \right|^2 \leq -\tau_-(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} \left| D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))] \right|^2 \, dt =$$

$$= -\tau_-(x,\xi) \int\limits_{\tau_-(x,\xi)}^{0} \sum_{\gamma \leq \beta} C_\gamma^\beta(x,\xi) \left| (D_z^\gamma F)(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t)) \right|^2 \, dt,$$

where $C_\gamma^\beta(x,\xi)$ are smooth functions. Integrating the last inequality, we obtain

$$\left\| \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))] \, dt \right\|_{L_2(U)}^2 \leq \sum_{\gamma \leq \beta} C_\gamma^\beta \int\limits_U \int\limits_{\tau_-(x,\xi)}^{0} |\tau_-(x,\xi)| \left| (D_z^\gamma F)(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t)) \right|^2 \, dt \, dy.$$

$$(3.3.17)$$

We change the integration variable in the integral on the right-hand side of (3.3.17) by the formula $z = G(x,\xi;t)$, where G is the mapping constructed in Corollary 3.3.3. By (3.3.10), after the change inequality (3.3.17) takes the form

$$\left\| \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))] \, dt \right\|_{L_2(U)}^2 \leq \sum_{\gamma \leq \beta} C_\gamma^\beta \int\limits_V \left| \frac{\tau_-(x,\xi)}{\langle \xi, \nu(x) \rangle} \right| \left| (D_z^\gamma F)(z) \right|^2 \, dz. \qquad (3.3.18)$$

By Lemma 3.2.2, the ratio $\tau_-(x,\xi)/\langle \xi, \nu(x) \rangle$ is bounded. Therefore (3.3.18) implies the desired estimate

$$\left\| \int\limits_{\tau_-(x,\xi)}^{0} D_y^\beta [F(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))] \, dt \right\|_{L_2(U)} \leq C \|F\|_{H^k(V)}.$$

Thus, the estimate (3.3.11) is proved for $f \in C^\infty(S^m \tau_M')$.

Let now $f \in H^k(S^m \tau_M')$. We define F as above, estimate (3.3.12) remaining valid. From the Fubini theorem we see that the integral on the right-hand side of equality (3.3.13) is finite for almost all $(x,\xi) \in \partial_+ \Omega M$; and the function If, defined by this equality, belongs to $H^0(\partial_+ \Omega M)$. We choose a sequence $f_\nu \in C^\infty(S^m \tau_M')$ $(\nu = 1, 2, \ldots)$ that converges to f in $H^k(S^m \tau_M')$. The sequence If_ν converges to If in $H^0(\partial_+ \Omega M)$. Applying estimate (3.3.11) for $f_\nu - f_\mu$, we see that If_ν is a Cauchy sequence in $H^k(\partial_+ \Omega M)$. Consequently, $If \in H^k(\partial_+ \Omega M)$ and estimate (3.3.11) is valid. The theorem is proved.

3.4 The problem of inverting the ray transform

Let M be a CDRM. Given a field $v \in C^\infty(S^{m-1} \tau_M')$ satisfying the boundary condition $v|_{\partial M} = 0$, equality (2.4.18) and definition (3.3.2) of the ray transform imply immediately that $I(dv) = 0$. From this, using Theorem 3.3.1 and boundedness of the trace operator $H^{k+1}(S^m \tau_M') \to H^k(S^m \tau_M'|_{\partial M})$, $v \mapsto v|_{\partial M}$, we obtain the next

Lemma 3.4.1 *Let M be a CDRM, $k \geq 0$ and $m \geq 0$ be integers. If a field $v \in H^{k+1}(S^m \tau_M')$ satisfies the boundary condition $v|_{\partial M} = 0$, then $Idv = 0$.*

By Theorem 2.4.2, a field $f \in H^k(S^m \tau_M')$ $(k \geq 1)$ can be uniquely decomposed into solenoidal and potential parts:

$$f = {}^s f + dv, \quad \delta\, {}^s f = 0, \quad v|_{\partial M} = 0, \qquad (3.4.1)$$

where ${}^s f \in H^k(S^m \tau_M')$ and $v \in H^{k+1}(S^{m-1} \tau_M')$. By Lemma 3.4.1, the ray transform pays no heed to the potential part of (3.4.1): $Idv = 0$. Consequently, given the ray transform If, we can hope to recover only the solenoidal part of the field f. We thus come to the next

Problem 3.4.2 (problem of inverting the ray transform) *For which CDRM can the solenoidal part of any field $f \in H^k(S^m \tau_M')$ be recovered from the ray transform If?*

The main result of the current section, Theorem 3.4.3 stated below, gives an answer for $k = 1$ under some assumption on the curvature of the manifold in the question. Let us now formulate the assumption.

Let M be a Riemannian manifold. Recall that, for a point $x \in M$ and a two-dimensional subspace $\sigma \subset T_x M$, by $K(x, \sigma)$ we denote the sectional curvature at the point x and in the two-dimensional direction σ which is defined by (2.3.6). For $(x, \xi) \in T^0 M$ we put

$$K(x, \xi) = \sup_{\sigma \ni \xi} K(x, \sigma), \qquad K^+(x, \xi) = \max\{0, K(x, \xi)\}. \tag{3.4.2}$$

For a CDRM (M, g), we introduce the next characteristic:

$$k^+(M, g) = \sup_{(x, \xi) \in \partial_- \Omega M} \int_0^{\tau_+(x, \xi)} t K^+(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)) \, dt. \tag{3.4.3}$$

We recall that here $\gamma_{x,\xi} : [0, \tau_+(x, \xi)] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \xi$. Note that $k^+(M, g)$ is a dimensionless quantity, i.e., it does not vary under multiplication of the metric g by a positive number.

Recall finally that, for $x \in \partial M$, we denote by $j_\nu : C^\infty(S^m \tau'_M|_{\partial M}) \to C^\infty(S^{m-1} \tau'_M|_{\partial M})$, the operator of contraction with the vector ν of the unit outer normal vector to the boundary.

We can now formulate our main result.

Theorem 3.4.3 *Let $n \geq 2$, $m \geq 0$ be integers, and (M, g) be a compact n-dimensional dissipative Riemannian manifold satisfying the condition*

$$k^+(M, g) < (n + 2m - 1)/m(m + n) \quad for \quad m > 0, \qquad k^+(M, g) < 1 \quad for \quad m = 0. \tag{3.4.4}$$

For every tensor field $f \in H^1(S^m \tau'_M)$, the solenoidal part ${}^s f$ is uniquely determined by the ray transform If and the next conditional stability estimate is valid:

$$\|{}^s f\|_0^2 \leq C \left(m \|j_\nu {}^s f|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2\right) \leq C_1 \left(m \|f\|_1 \cdot \|If\|_0 + \|If\|_1^2\right) \tag{3.4.5}$$

where constants C and C_1 are independent of f.

We will make a few remarks on the theorem.

The first summand on the right-hand side of estimate (3.4.5) shows that the problem of recovering ${}^s f$ from If is perhaps of conditionally-correct nature: for stably determining ${}^s f$, we are to have an a priori estimate for $\|f\|_1$. Note that this summand has appeared due to the method applied in our proof; the author knows nothing about any example demonstrating that the problem is conditionally-correct as a matter of fact. The factor m before the first summand is distinguished so as to emphasize that in the case $m = 0$ the problem is correct.

In order to avoid complicated formulations and proofs, in the current and previous lectures we use the Sobolev spaces H^k only for integral $k \geq 0$. If the reader is familiar with the definition of these spaces for fractional k, he/she can verify, by examining the proof below, that it is possible to replace the factor $\|f\|_1$ in (3.4.5) by $\|f\|_{1/2}$.

We emphasize that (3.4.4) is a restriction only on the positive values of the sectional curvature, which is of an integral nature, moreover.

The right-hand side of equality (3.4.4) takes its maximal value for $m = 0$. If a CDRM (M, g) satisfies the condition

$$k^+(M, g) < 1, \tag{3.4.6}$$

then the next claims are valid: 1) M is diffeomorphic to the ball, and 2) the metric g is simple in the sense of the definition given in Section 3.1. We will not give here the proof, of the claims, which is beyond the scope of our lectures (and will not use these claims). We will only discuss briefly a possible way of the proof. First of all, condition (3.4.6) implies absence of conjugate points. This fact can be proved as follows: first, by arguments similar to those used in the proof of the theorem on comparing indices [33], we reduce the question to the two-dimensional case; then applying the Hartman-Wintner theorem (Theorem 5.1 of [37]). With the absence of conjugate points available, our claims can be established by arguments similar to those used in the proof of the Hadamard-Cartan theorem [33].

The proof of Theorem 3.4.3 will be given in the next lecture after developing some techniques. Now we will show that this theorem follows from the next special case of it.

Lemma 3.4.4 *Let a CDRM (M, g) satisfies (3.4.4). For a field $f \in C^\infty(S^m \tau'_M)$ satisfying the condition*

$$\delta f = 0, \tag{3.4.7}$$

the estimate

$$\|f\|_0^2 \leq C \left(m \|j_\nu f|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2 \right) \tag{3.4.8}$$

holds with a constant C independent of f.

Indeed, given a field $f \in H^1(S^m \tau'_M)$, let

$$f = {}^s f + dv, \quad \delta\, {}^s f = 0, \quad v|_{\partial M} = 0 \tag{3.4.9}$$

be the decomposition into the solenoidal and potential parts, where ${}^s f \in H^1(S^m \tau'_M)$ and $v \in H^2(S^{m-1}\tau'_M)$. By Theorem 2.4.2, the estimate

$$\|{}^s f\|_1 \leq C_1 \|f\|_1 \tag{3.4.10}$$

holds. We choose a sequence, of fields $f_k \in C^\infty(S^m \tau'_M)$ $(k = 1, 2, \ldots)$, which converges to f in $H^1(S^m \tau'_M)$. Applying Theorem 2.4.2 to f_k, we obtain the decomposition

$$f_k = {}^s f_k + dv_k, \quad \delta\, {}^s f_k = 0, \quad v_k|_{\partial M} = 0 \tag{3.4.11}$$

with ${}^s f_k \in C^\infty(S^m \tau'_M)$, $v_k \in C^\infty(S^{m-1}\tau'_M)$. Since ${}^s f$ in (3.4.9) depends continuously on f, as have been shown in Theorem 2.4.2,

$${}^s f_k \rightarrow {}^s f \quad \text{in} \quad H^1(S^m \tau'_M) \quad \text{as} \quad k \rightarrow \infty. \tag{3.4.12}$$

In the view of boundedness of the trace operator $H^1(S^m \tau'_M) \rightarrow H^0(S^m \tau'_M|_{\partial M})$, (3.4.12) implies that

$${}^s f_k|_{\partial M} \rightarrow {}^s f|_{\partial M} \quad \text{in} \quad H^0(S^m \tau'_M|_{\partial M}) \quad \text{as} \quad k \rightarrow \infty. \tag{3.4.13}$$

By Lemma 3.4.1, the equalities $v_k|_{\partial M} = 0$ and $v|_{\partial M} = 0$ imply that $I(dv_k) = I(dv) = 0$. Therefore, from (3.4.9) and (3.4.11), we obtain

$$If = I\, {}^s f, \quad If_k = I\, {}^s f_k. \tag{3.4.14}$$

Applying Lemma 3.4.4 to ${}^s f_k$, we have

$$\|{}^s f_k\|_0^2 \leq C \left(m \|j_\nu\, {}^s f_k|_{\partial M}\|_0 \cdot \|I\, {}^s f_k\|_0 + \|I\, {}^s f_k\|_1^2 \right).$$

By (3.4.14), the last inequality can be rewritten as:

$$\|{}^s f_k\|_0^2 \leq C \left(m \|j_\nu\, {}^s f_k|_{\partial M}\|_0 \cdot \|If_k\|_0 + \|If_k\|_1^2 \right).$$

We pass to the limit in this inequality as $k \rightarrow \infty$; and make use of (3.4.11), (3.4.12) and continuity of I proved in Theorem 3.3.1. In such a way we arrive at the estimate

$$\|{}^s f\|_0^2 \leq C \left(m \|j_\nu\, {}^s f|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2 \right). \tag{3.4.15}$$

Using (3.4.10) and continuity of the trace operator $H^1(S^m \tau'_M) \rightarrow H^0(S^m \tau'_M|_{\partial M})$, we obtain

$$\|j_\nu\, {}^s f|_{\partial M}\|_0 \leq C_2 \|{}^s f|_{\partial M}\|_0 \leq C_3 \|{}^s f\|_1 \leq C_4 \|f\|_1. \tag{3.4.16}$$

Inequalities (3.4.15) and (3.4.16) give the claim of Theorem 3.4.3.

3.5 The kinetic equation on a Riemannian manifold

In this section we reduce Theorem 3.4.3 to an inverse problem for a differential equation on the manifold ΩM.

Let a field $f \in C^\infty(S^m \tau'_M)$ on a CDRM M satisfy the conditions of Lemma 3.4.4. We define the function

$$u(x, \xi) = \int_{\tau_-(x, \xi)}^{0} \langle f(\gamma_{x,\xi}(t)), \dot{\gamma}_{x,\xi}^m(t) \rangle \, dt \qquad ((x, \xi) \in T^0 M) \tag{3.5.1}$$

on T^0M, using the same notation as used in definition (3.3.2) of the ray transform. The difference between equalities (3.3.2) and (3.5.1) is the fact that the first of them is considered only for $(x, \xi) \in \partial_+ \Omega M$ while the second one, for all $(x, \xi) \in T^0M$. In particular, we have the boundary condition

$$u|_{\partial_+ \Omega M} = If. \tag{3.5.2}$$

Since $\tau_-(x, \xi) = 0$ for $(x, \xi) \in \partial_- \Omega M$, we have the second boundary condition

$$u|_{\partial_- \Omega M} = 0. \tag{3.5.3}$$

The function $u(x, \xi)$ is smooth at the same points at which $\tau_-(x, \xi)$ is smooth. The last is true, as we know, at all points of the open set $T^0M \setminus T(\partial M)$ of the manifold T^0M.

Let us show that the function u satisfies the equation

$$Hu = f_{i_1 \ldots i_m}(x)\xi^{i_1} \ldots \xi^{i_m} \tag{3.5.4}$$

on $T^0M \setminus T(\partial M)$, where $H \in C^\infty(\tau_{TM})$ is the geodesic vector field on TM defined in coordinates by formula (2.6.1).

Indeed, let $(x, \xi) \in T^0M \setminus T(\partial M)$ and $\gamma = \gamma_{x,\xi} : [\tau_-(x, \xi), \tau_+(x, \xi)] \to M$ be the geodesic defined by the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. For sufficiently small $s \in \mathbf{R}$, we put $x_s = \gamma(s)$ and $\xi_s = \dot\gamma(s)$. Then $\gamma_{x_s, \xi_s}(t) = \gamma(t + s)$ and $\tau_-(x_s, \xi_s) = \tau_-(x, \xi) - s$. Consequently,

$$u(\gamma(s), \dot\gamma(s)) = u(x_s, \xi_s) = \int\limits_{\tau_-(x_s,\xi_s)}^{0} \langle f(\gamma_{x_s,\xi_s}(t)), \dot\gamma^m_{x_s,\xi_s}(t)\rangle \, dt = \int\limits_{\tau_-(x,\xi)}^{s} \langle f(\gamma_{x,\xi}(t)), \dot\gamma^m_{x,\xi}(t)\rangle \, dt.$$

Differentiating this equality with respect to s and putting $s = 0$ in the so-obtained relation, we come to

$$\dot\gamma^i(0)\frac{\partial u}{\partial x^i} + \ddot\gamma^i(0)\frac{\partial u}{\partial \xi^i} = f_{i_1 \ldots i_m}(\gamma(0))\dot\gamma^{i_1}(0) \ldots \dot\gamma^{i_m}(0). \tag{3.5.5}$$

Inserting $\gamma(0) = x$, $\dot\gamma(0) = \xi$ and the value $\ddot\gamma^i(0) = -\Gamma^i_{jk}(x)\xi^j \xi^k$ from equation (2.3.5) of geodesics into the last relation and taking (2.6.1) into account, we arrive at (3.5.4).

The function $u(x, \xi)$ is positively homogeneous in its second argument:

$$u(x, \lambda\xi) = \lambda^{m-1}u(x, \xi) \qquad (\lambda > 0). \tag{3.5.6}$$

Indeed, since $\gamma_{x,\lambda\xi}(t) = \gamma_{x,\xi}(\lambda t)$, it follows from (3.5.1) that

$$u(x, \lambda\xi) = \int\limits_{\tau_-(x,\lambda\xi)}^{0} \langle f(\gamma_{x,\lambda\xi}(t)), \dot\gamma^m_{x,\lambda\xi}(t)\rangle \, dt = \int\limits_{\lambda^{-1}\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(\lambda t)), \lambda^m\dot\gamma^m_{x,\xi}(\lambda t)\rangle \, dt =$$

$$= \lambda^{m-1} \int\limits_{\tau_-(x,\xi)}^{0} \langle f(\gamma_{x,\xi}(t)), \dot\gamma^m_{x,\xi}(t)\rangle \, dt = \lambda^{m-1}u(x, \xi).$$

Thus, the function $u(x, \xi)$ is a solution to the boundary value problem (3.5.2)–(3.5.4) and satisfies the homogeneity condition (3.5.6). Besides, we recall that condition (3.4.7) is imposed upon the field f on the right-hand side of equation (3.5.4). Lemma 3.4.4 thereby reduces to the next problem: one has to estimate the right-hand side of equation (3.5.4) by the right-hand side of the boundary condition (3.5.2).

The manifold ΩM is invariant with respect to the geodesic flow. This means that the field H is tangent to ΩM at all points of the manifold ΩM and, consequently, equation (3.5.4) can be considered on ΩM.

The operator H is related to the inner differentiation operator d by the following equality:

$$H\left(v_{i_1 \ldots i_{m-1}}(x)\xi^{i_1} \ldots \xi^{i_{m-1}}\right) = (dv)_{i_1 \ldots i_m}(x)\xi^{i_1} \ldots \xi^{i_m}, \tag{3.5.7}$$

which can be proved by an easy calculation in coordinates.

The equation

$$Hu = F(x, \xi) \tag{3.5.8}$$

on ΩM, with the right-hand side depending arbitrarily on ξ, is called (*stationary, unit-velocity*) *kinetic equation of the metric* g. It has a simple physical sense. Let us imagine a stationary distribution of particles moving in M. Every particle moves along a geodesic of the metric g with unit speed, the particles do not influence one another and the medium. Assume that there are also sources of particles in M. By $u(x, \xi)$ and $F(x, \xi)$ we mean the densities of particles and sources with respect to the volume form $d\Sigma^{2n-1}$ defined by (2.7.18). Then equation (3.5.8) is valid. We omit its proof which can be done in exact analogy with the proof of the Liouville theorem well-known in statistical physics [87].

If the source $F(x, \xi)$ is known then, to get a unique solution u to equation (3.5.8), one has to set the incoming flow $u|_{\partial_- \Omega M}$. In particular, the first boundary conditions (3.5.3) means the absence of the incoming flow. The second boundary conditions (3.5.2), i.e., the outgoing flow $u|_{\partial_+ \Omega M}$, must be used for the inverse problem of determining the source. This inverse problem has the very essential (and not although quite physical) requirement on the source to depend polynomially on the direction ξ. The operator d gives us the next means of constructing sources which are invisible from outside and polynomial in ξ: if $v \in C^\infty(S^{m-1}\tau'_M)$ and $v|_{\partial M} = 0$, then the source $F(x, \xi) = (dv)_{i_1 \ldots i_m}\xi^{i_1} \ldots \xi^{i_m}$ is invisible from outside. Does this construction exhaust all sources that are invisible from outside and polynomial in ξ? It is the physical interpretation of Problem 3.4.2.

3.6   Some remarks

The boundary rigidity problem was first posed explicitly by R. Michel [50]. Nevertheless, for some special classes of metrics, the problem was considered before in mathematical geophysics, as we have described in Section 1.3. We will now list the known results on Problem 3.1.1. In [9] Yu. E. Anikonov has proved an assertion that amounts to the following: a simple Riemannian metric on a compact two-dimensional manifold M is flat if and only if any geodesic triangle with vertices on ∂M has the sum of angles equal to π. As is easily seen these angles can be expressed by the boundary distance function. Thus, this result answers the question: how to determine whether a metric is flat given the boundary distance function? A similar result was obtained by M. L. Gerver and N. S. Nadirashvili [29]. R. Michel obtained a positive answer to Problem 3.1.1 in the two-dimensional case when one of the two given metrics has the constant Gauss curvature [50]. A positive answer to Problem 3.1.1 for a rather wide class of two-dimensional metrics has been obtained in [30]. C. B. Croke [18] and J.-P. Otal [64] solved Problem 3.1.1 for two-dimensional manifolds of nonpositive curvature satisfying the following condition: the length of every geodesic is equal to the distance between its endpoints. It is called the SGM-condition (segment geodesic minimizing), and generalizes the condition of simplicity of the metric.

We see that all above mentioned results are dealing with the two-dimensional case. The first and, as the author knows, the only result in the multidimensional case has been obtained by M. Gromov [34]. He has found a positive answer to Problem 3.1.1 under the assumption of flatness of one of the two metrics. A simple proof of this result is presented in [19]. This result can also be derived from the Hopf conjecture. The multidimensional version of the Hopf conjecture was proved by D. Burago and S. Ivanov [16].

We obtain the important special case of Problem 3.1.1 assuming the metrics under consideration to be conformally equivalent. In this case the problem was solved in [58, 14, 19].

In the case of $m = 0$ a solution to linear Problem 3.1.2 for simple metrics was found by R. G. Mukhometov [55, 58], I. N. Bernstein and M. L. Gerver [14], and in the case of $m = 1$, by Yu. E. Anikonov and V. G. Romanov [8, 10]. In Lecture 6 we will give an alternative proof of these results. For $m \geq 2$ no result like these has been obtained until now.

In the case of $M = \mathbf{R}^n$ with the Euclidean metric, the ray transform of a compactly supported field $f \in C^\infty(S^m\tau'_{\mathbf{R}^n})$ can be written in the form

$$If(x, \xi) = \int\limits_{-\infty}^{\infty} f_{i_1 \ldots i_m}(x + t\xi)\xi^{i_1} \ldots \xi^{i_n}\, dt \quad (x \in \mathbf{R}^n, 0 \neq \xi \in \mathbf{R}^n).$$

The detailed theory of this transform is presented in Chapter 2 of [77]. In particular, there is an explicit inversion formula of Radon's type expressing the solenoidal part of a field f through If.

Lecture 4
Inversion of the ray transform

Here we finish the proof of Theorem 3.4.3.

Sections 4.1–4.2 contain three auxiliary claims which are used in the proof of Theorem 3.4.3. Two of them, the Pestov identity and the Poincaré inequality for semibasic tensor fields have certain significance besides the proof of the theorem; use of them will be made in the next lectures.

In Sections 4.3 we present the proof of Theorem 3.4.3. Some quadratic integral identity is proved for the kinetic equation. For the summands of the last identity, some estimates are obtained which lead to the claim of the theorem.

In Section 4.4 we present some alternative approach to Problem 3.4.2. Till now this approach is realized only in the two-dimensional case for tensor fields of second degree.

4.1   Pestov's differential identity

Recall that in Lecture 2 we introduced the space $C^\infty(\beta_s^r M)$ of semibasic tensor fields of degree (r, s) over the space TM of the tangent bundle of a Riemannian manifold (M, g) and defined the operators $\overset{v}{\nabla}, \overset{h}{\nabla} : C^\infty(\beta_s^r M) \to C^\infty(\beta_{s+1}^r M)$ of vertical and horizontal differentiation. The metric g establishes the canonical isomorphism of the bundles $\beta_s^r M \cong \beta_0^{r+s} M \cong \beta_{r+s}^0 M$; in coordinate form this fact is expressed by the known operations of raising and lowering indices of a tensor; we will use them everywhere. Similar notation will be used for the derivative operators: $\overset{v}{\nabla}{}^i = g^{ij}\overset{v}{\nabla}_j$, $\overset{h}{\nabla}{}^i = g^{ij}\overset{h}{\nabla}_j$.

The metric g allows us to introduce the inner product on the bundle $\beta_s^r M$. Consequently, for $u, v \in C^\infty(\beta_m^0 M)$, the inner product $\langle u, v \rangle$ is a function on TM expressible in coordinate form as

$$\langle u(x, \xi), v(x, \xi) \rangle = u_{i_1 \ldots i_m}(x, \xi) v^{i_1 \ldots i_m}(x, \xi). \tag{4.1.1}$$

We also denote $|u(x, \xi)|^2 = \langle u(x, \xi), u(x, \xi) \rangle$. The notations $\langle u(x, \xi), v(x, \xi) \rangle$ and $|u(x, \xi)|^2$ can be considered as convenient abbreviations of the functions on the right-hand side of (4.1.1), and we will make wide use of them.

The following statement is a multidimensional analog of Lemma 1.1.2.

Lemma 4.1.1 (the Pestov identity) *Let M be a Riemannian manifold. For a function $u \in C^\infty(TM)$, the next identity is valid on TM:*

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu) \rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - R_{ijkl}\xi^i \xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u, \tag{4.1.2}$$

where the semibasic vector fields v and w are defined by the equalities

$$v^i = \xi^i \overset{h}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j u - \xi^j \overset{v}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j u, \tag{4.1.3}$$

$$w^i = \xi^j \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j u. \tag{4.1.4}$$

Proof. From the definition of the operator H, we have

$$2\overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i \left(\xi^j \overset{h}{\nabla}_j u \right).$$

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Using the relation $\overset{v}{\nabla}_i \xi^j = \delta_i^j$, we obtain

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_i u + 2\xi^j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u. \tag{4.1.5}$$

We transform the second summand on the right-hand side of the last relation. To this end we define a function φ by the equality

$$2\xi^j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u = \overset{v}{\nabla}_i\left(\xi^j \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j u\right) + \overset{h}{\nabla}_j\left(\xi^j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i u\right) - \overset{h}{\nabla}{}^i\left(\xi^j \overset{v}{\nabla}_i u \cdot \overset{h}{\nabla}_j u\right) - \varphi. \tag{4.1.6}$$

Let us show that φ is independent of second-order derivatives of the function u. Indeed, expressing the derivatives of the products on the right-hand side of (4.1.6) through the derivatives of the factors, we obtain

$$\varphi = -2\xi^j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u + \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_i u + \xi^j \overset{v}{\nabla}_i \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j u + \xi^j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u +$$

$$+ \xi^j \overset{h}{\nabla}_j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i u + \xi^j \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_j \overset{v}{\nabla}_i u - \xi^j \overset{h}{\nabla}{}^i \overset{v}{\nabla}_i u \cdot \overset{h}{\nabla}_j u - \xi^j \overset{v}{\nabla}_i u \cdot \overset{h}{\nabla}{}^i \overset{h}{\nabla}_j u.$$

After evident transformations, this equality takes the form

$$\varphi = \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_i u + \xi^j \overset{h}{\nabla}{}^i u \cdot \left(\overset{h}{\nabla}_j \overset{v}{\nabla}_i - \overset{v}{\nabla}_i \overset{h}{\nabla}_j\right) u + \xi^j \overset{h}{\nabla}_j u \cdot \left(\overset{v}{\nabla}_i \overset{h}{\nabla}{}^i - \overset{h}{\nabla}{}^i \overset{v}{\nabla}_i\right) u + \xi^j \overset{v}{\nabla}{}^i u \cdot \left(\overset{h}{\nabla}_j \overset{h}{\nabla}_i - \overset{h}{\nabla}_i \overset{h}{\nabla}_j\right) u.$$

Using commutation formulas for the operators $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$ which are presented in Theorem 2.6.2, we obtain

$$\varphi = \overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_i u - \xi^j \overset{v}{\nabla}{}^i u \cdot R^p_{qji}\xi^q \overset{v}{\nabla}_p u.$$

Inserting this expression for the function φ into (4.1.6), we have

$$2\xi^j \overset{h}{\nabla}{}^i u \cdot \overset{v}{\nabla}_i \overset{h}{\nabla}_j u = -\overset{h}{\nabla}{}^i u \cdot \overset{h}{\nabla}_i u + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - R_{ijkl}\xi^i \xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u.$$

Finally, replacing the second summand on the right-hand side of (4.1.5) with the last value, we arrive at (4.1.2). The lemma is proved.

4.2   Poincaré's inequality for semibasic tensor fields

Lemma 4.2.1 *Let M be a CDRM and λ be a continuous nonnegative function on ΩM. For a semibasic tensor field $f \in C^\infty(\beta_m^0 M)$ satisfying the boundary condition*

$$f|_{\partial_- \Omega M} = 0, \tag{4.2.1}$$

the next inequality is valid:

$$\int\limits_{\Omega M} \lambda(x,\xi)|f(x,\xi)|^2\, d\Sigma \leq \lambda_0 \int\limits_{\Omega M} |Hf(x,\xi)|^2\, d\Sigma, \tag{4.2.2}$$

where

$$\lambda_0 = \sup_{(x,\xi)\in\partial_- \Omega M} \int\limits_0^{\tau_+(x,\xi)} t\lambda(\gamma_{x,\xi}(t), \dot\gamma_{x,\xi}(t))\, dt, \tag{4.2.3}$$

$\gamma_{x,\xi} : [0, \tau_+(x,\xi)] \to M$ *is a maximal geodesic defined by the initial conditions* $\gamma_{x,\xi}(0) = x$ *and* $\dot\gamma_{x,\xi}(0) = \xi$, $d\Sigma = d\Sigma^{2n-1}$ *is the volume form on ΩM defined by formula (2.7.18).*

By the Liouville theorem [15], the geodesic flow preserves the volume form $d\Sigma$. Therefore, in the scalar case $f = \varphi \in C^\infty(\beta_0^0 M)$, the lemma coincides, in fact, with the well-known Poincaré inequality [54]. The case of an arbitrary semibasic field $f \in C^\infty(\beta_m^0 M)$ is reduced to the scalar one by introducing the function $\varphi(x,\xi) = |f(x,\xi)|$. Unfortunately, in such a way an additional obstacle arises that relates to the singularities of the function φ at zeros of the field f. For this reason we should reproduce the proof of the Poincaré inequality, while taking the nature of the mentioned singularities into account. First of all we will reduce Lemma 4.2.1 to the next claim:

Lemma 4.2.2 *Let M, λ and λ_0 be the same as in Lemma 4.2.1; a function $\varphi \in C(\Omega M)$ be smooth on $\Omega_\varphi = \{(x, \xi) \in \Omega M \mid \varphi(x, \xi) \neq 0\}$. Suppose that*

$$\sup_{(x,\xi) \in \Omega_\varphi} |H\varphi(x, \xi)| < \infty. \tag{4.2.4}$$

If φ satisfies the boundary condition

$$\varphi|_{\partial_- \Omega M} = 0, \tag{4.2.5}$$

then the next estimate is valid:

$$\int\limits_{\Omega M} \lambda(x, \xi) |\varphi(x, \xi)|^2 \, d\Sigma \leq \lambda_0 \int\limits_{\Omega_\varphi} |H\varphi(x, \xi)|^2 \, d\Sigma. \tag{4.2.6}$$

Proof of Lemma 4.2.1. Let a semibasic field f satisfy the conditions of Lemma 4.2.1. We verify that the function $\varphi = |f|$ satisfies the conditions of Lemma 4.2.2. The only nontrivial condition is (4.2.4). The equality $H\varphi = \langle f, Hf \rangle / |f|$ holds on Ω_φ. It implies that $|H\varphi| \leq |Hf|$ and, consequently, (4.2.4) holds.

Assuming validity of Lemma 4.2.2, we have inequality (4.2.6). From this inequality we obtain

$$\int\limits_{\Omega M} \lambda |f|^2 \, d\Sigma = \int\limits_{\Omega M} \lambda |\varphi|^2 \, d\Sigma \leq \lambda_0 \int\limits_{\Omega_\varphi} |H\varphi|^2 \, d\Sigma =$$

$$= \lambda_0 \int\limits_{\Omega_\varphi} \frac{|\langle f, Hf \rangle|^2}{|f|^2} \, d\Sigma \leq \lambda_0 \int\limits_{\Omega_\varphi} |Hf|^2 \, d\Sigma \leq \lambda_0 \int\limits_{\Omega M} |Hf|^2 \, d\Sigma.$$

The lemma is proved.

Proof of Lemma 4.2.2. With the help of the Santalo formula (Lemma 3.3.2), inequality (4.2.6) can be rewritten in the form

$$\int\limits_{\partial_- \Omega M} \langle \xi, -\nu(x) \rangle d\Sigma^{2n-2}(x, \xi) \int\limits_0^{\tau_+(x,\xi)} \rho(x, \xi; t) |\psi(x, \xi; t)|^2 dt \leq \lambda_0 \int\limits_{D_\psi} \langle \xi, -\nu(x) \rangle \left| \frac{\partial \psi(x, \xi; t)}{\partial t} \right|^2 dt \, d\Sigma^{2n-2}(x, \xi), \tag{4.2.7}$$

where $\rho(x, \xi; t) = \lambda(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))$, $\psi(x, \xi; t) = \varphi(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))$, and $D_\psi = \{(x, \xi; t) \mid \psi(x, \xi; t) \neq 0\} \subset \partial_- \Omega M \times \mathbf{R}$. The function ψ is continuous on $D = \{(x, \xi; t) \mid 0 \leq t \leq \tau_+(x, \xi)\} \subset \partial_- \Omega M \times \mathbf{R}$, smooth on D_ψ and, by (4.2.4) and (4.2.5), satisfies the conditions

$$\sup_{D_\psi} |\partial \psi(x, \xi; t)/\partial t| < \infty, \tag{4.2.8}$$

$$\psi(x, \xi; 0) = 0. \tag{4.2.9}$$

The constant λ_0 of Lemma 4.2.1 is expressed through ρ:

$$\lambda_0 = \sup_{(x,\xi) \in \partial_- \Omega M} \int\limits_0^{\tau_+(x,\xi)} t\rho(x, \xi; t) \, dt. \tag{4.2.10}$$

We define the function $\dot{\psi} : D \to \mathbf{R}$ by putting

$$\dot{\psi}(x, \xi; t) = \begin{cases} \partial \psi(x, \xi; t)/\partial t & \text{for} \quad (x, \xi; t) \in D_\psi, \\ 0 & \text{for} \quad (x, \xi; t) \notin D_\psi. \end{cases} \tag{4.2.11}$$

To prove (4.2.7), it suffices to show that

$$\int\limits_0^{\tau_+(x,\xi)} \rho(x, \xi; t) |\psi(x, \xi; t)|^2 dt \leq \lambda_0 \int\limits_0^{\tau_+(x,\xi)} |\dot{\psi}(x, \xi; t)|^2 dt \tag{4.2.12}$$

for every $(x, \xi) \in \partial_- \Omega M$.

Let us consider the function $\psi_y(t) = \psi(x, \xi; t)$, for a fixed $y = (x, \xi)$, as a function in the variable $t \in I_y = (0, \tau_+(y))$. We shall prove that it is absolutely continuous on I_y. Indeed, let $J_y = \{t \in I_y \mid \psi_y(t) \neq 0\}$. As an open subset of I_y, the set J_y is a union of pairwise disjoint intervals $J_y = \bigcup_{i=1}^{\infty}(a_i, b_i)$. The function ψ_y is smooth on each of these intervals and, by (4.2.8), its derivative is bounded:

$$|d\psi_y(t)/dt| \leq C \qquad (t \in (a_i, b_i)), \tag{4.2.13}$$

where a constant C is the same for all i. The function ψ_y vanishes on $I_y \setminus J_y$ and is continuous on I_y. The listed properties imply that

$$|\psi_y(t_1) - \psi_y(t_2)| \leq C|t_1 - t_2| \tag{4.2.14}$$

for all $t_1, t_2 \in I_y$. In particular, (4.2.14) implies absolute continuity of ψ_y. Consequently, this function is differentiable almost everywhere on I_y and can be recovered from its derivative:

$$\psi_y(t) = \int\limits_0^t \frac{d\psi_y(\tau)}{d\tau}\, d\tau. \tag{4.2.15}$$

While writing down the last equality, we took (4.2.9) into account. The derivative $d\psi_y(t)/dt$ is bounded. From (4.2.15) with the help of the Cauchy-Bunyakovskiĭ inequality, we obtain

$$|\psi_y(t)|^2 \leq t \int\limits_0^t \left|\frac{d\psi_y(\tau)}{d\tau}\right|^2 d\tau \leq t \int\limits_0^{\tau_+(y)} \left|\frac{d\psi_y(t)}{dt}\right|^2 dt. \tag{4.2.16}$$

Let us show that, almost everywhere on I_y, $d\psi_y(t)/dt$ coincides with the function $\dot\psi(y; t)$ defined by formula (4.2.11). Indeed, by (4.2.11), $d\psi_y(t)/dt = \dot\psi(y; t)$ if $t \in J_y$. If $t \in I_y \setminus J_y$ does not coincide with any of the endpoints of the intervals (a_i, b_i), then t is a limit point of the set $I_y \setminus J_y$. Since $\psi_y|_{I_y \setminus J_y} = 0$, existence of the derivative $d\psi_y(t)/dt$ implies that it is equal to zero. The function $\dot\psi(y; t)$ vanishes on $I_y \setminus J_y$, by definition (4.2.11). Thus the relation $d\psi_y(t)/dt = \dot\psi(y; t)$ is proved for all $t \in I_y$ such that the derivative $d\psi_y(t)/dt$ exists, with the possible exception of the endpoints of the intervals (a_i, b_i).

We can now rewrite (4.2.16) as:

$$|\psi(x, \xi; t)|^2 \leq t \int\limits_0^{\tau_+(x,\xi)} |\dot\psi(x, \xi; t)|^2 dt.$$

We multiply this inequality by $\rho(x, \xi; t)$ and integrate it with respect to t

$$\int\limits_0^{\tau_+(x,\xi)} \rho(x, \xi; t)|\psi(x, \xi; t)|^2 dt \leq \int\limits_0^{\tau_+(x,\xi)} t\rho(x, \xi; t)\, dt \int\limits_0^{\tau_+(x,\xi)} |\dot\psi(x, \xi; t)|^2 dt.$$

By (4.2.10), the first integral on the right-hand side of the last formula can be replaced by λ_0. We thus arrive at (4.2.12). The lemma is proved.

To prove Lemma 3.4.4 we need also the next claim. It is of a purely algebraic nature, although formulated in terms of analysis.

Lemma 4.2.3 *Let M be a compact n-dimensional Riemannian manifold, $f \in C^\infty(S^m \tau_M')$, $m \geq 1$. Define the function $\varphi \in C^\infty(TM) = C^\infty(\beta_0^0 M)$ and semibasic covector field $F \in C^\infty(\beta_1^0 M)$ by the equalities*

$$\varphi(x, \xi) = f_{i_1 \ldots i_m}(x)\xi^{i_1} \ldots \xi^{i_m}; \qquad F_i(x, \xi) = f_{i i_2 \ldots i_m}(x)\xi^{i_2} \ldots \xi^{i_m}. \tag{4.2.17}$$

Then the next inequality is valid:

$$\int\limits_{\Omega M} |F|^2 d\Sigma \leq \frac{n + 2m - 2}{m} \int\limits_{\Omega M} |\varphi|^2 d\Sigma. \tag{4.2.18}$$

Proof. We shall show that this claim is reduced to a known property of eigenvalues of the Laplacian on the sphere.

It follows from (2.7.29) that inequality (4.2.18) is equivalent to the next one:

$$\int\limits_{M} \left[\int\limits_{\Omega_x M} |F(x,\xi)|^2 d\omega_x(\xi) \right] dV^n(x) \le \frac{n+2m-2}{m} \int\limits_{M} \left[\int\limits_{\Omega_x M} |\varphi(x,\xi)|^2 d\omega_x(\xi) \right] dV^n(x).$$

Consequently, to prove the lemma it suffices to show that

$$\int\limits_{\Omega_x M} |F(x,\xi)|^2 d\omega_x(\xi) \le \frac{n+2m-2}{m} \int\limits_{\Omega_x M} |\varphi(x,\xi)|^2 d\omega_x(\xi). \tag{4.2.19}$$

for every $x \in M$.

Fixing a point x, we introduce coordinates in some of its neighborhoods so that $g_{ij}(x) = \delta_{ij}$. With the help of these coordinates we identify $T_x M$ and $\mathbf{R}^n$, the latter furnished with the standard Euclidean metric. Then $\Omega_x M$ is identified with the unit sphere Ω of the space $\mathbf{R}^n$; the measure $d\omega_x$, with the standard angle measure $d\omega$; the function $\varphi(x,\xi)$, by (4.2.17), with a homogeneous polynomial ψ of degree m on $\mathbf{R}^n$; the field F, with $\nabla\psi/m$. Thus, to prove (4.2.19) it suffices to verify the inequality

$$\int\limits_{\Omega} |\nabla\psi|^2 d\omega \le m(n+2m-2) \int\limits_{\Omega} |\psi|^2 d\omega \tag{4.2.20}$$

on the space $P_m(\mathbf{R}^n)$ of homogeneous polynomials of degree m. Applying the Green formula

$$\int\limits_{\Omega} |\nabla\psi|^2 d\omega = \int\limits_{\Omega} \psi(m^2 - \Delta_\omega)\, \psi\, d\omega \qquad (\psi \in P_m(\mathbf{R}^n)),$$

where Δ_ω is the spherical Laplacian [83], we see that (4.2.20) is equivalent to the claim: all eigenvalues μ_k of the operator $m^2 - \Delta_\omega$ on the space $P_m(\mathbf{R}^n)$ do not exceed $m(n+2m-2)$. It is known that the eigenvalues of the Laplacian Δ_ω are precisely the numbers $\lambda_k = -k(n+k-2)$ and the spherical harmonics of order k are just the eigenfunctions belonging to λ_k. Therefore the eigenvalues of the operator $m^2 - \Delta_\omega$ on $P_m(\mathbf{R}^n)$ are those of $\mu_k = m^2 - \lambda_k$ for which $k \le m$. The maximal of them is $m(n+2m-2)$. The lemma is proved.

4.3 Proof of Theorem 3.4.3

To prove Theorem 3.4.3 it suffices to prove Lemma 3.4.4.

Let a field $f \in C^\infty(S^m \tau'_M)$ on a CDRM M satisfy the conditions of Lemma 3.4.4. We define the function $u(x,\xi)$ for $(x,\xi) \in T^0 M$ by formula (3.5.1). This function is continuous on $T^0 M$, smooth on $T^0 M \setminus T(\partial M)$, satisfies the kinetic equation (3.5.4) on $T^0 M \setminus T(\partial M)$, boundary conditions (3.5.2) and (3.5.3), and the homogeneity condition (3.5.6).

The Pestov identity (4.1.2) is valid for the function $u(x,\xi)$ on $T^0 M \setminus T(\partial M)$. Using (3.5.4), we transform the left-hand side of the identity

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2\overset{h}{\nabla}^i u \cdot \overset{v}{\nabla}_i(Hu) = 2\overset{h}{\nabla}^i u \cdot \frac{\partial}{\partial \xi^i}\left(f_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}\right) =$$

$$= 2m\overset{h}{\nabla}^i u \cdot f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} = \overset{h}{\nabla}^i(2mu f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) - 2mu(\overset{h}{\nabla}^i f_{i i_2\dots i_m})\xi^{i_2}\dots\xi^{i_m} =$$

$$= \overset{h}{\nabla}_i \tilde{v}^i - 2mu(\delta f)_{i_1\dots i_{m-1}}\xi^{i_1}\dots\xi^{i_{m-1}}, \tag{4.3.1}$$

where

$$\tilde{v}^i = 2mu\, g^{ip} f_{p i_1\dots i_{m-1}}\xi^{i_1}\dots\xi^{i_{m-1}}. \tag{4.3.2}$$

The second summand on the right-hand side of (4.3.1) vanishes by (3.4.7), and we obtain

$$2\langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = \overset{h}{\nabla}_i \tilde{v}^i. \tag{4.3.3}$$

. Thus, the application of Lemma 4.1.1 to function (3.5.1) leads to the next identity on $T^0 M \setminus T(\partial M)$:

$$|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u = \overset{h}{\nabla}_i(\tilde{v}^i - v^i) - \overset{v}{\nabla}_i w^i, \qquad (4.3.4)$$

where the semibasic vector fields v, w and $\tilde{v}$ are defined by formulas (4.1.3), (4.1.4) and (4.3.2).

We are going to integrate equality (4.3.4) over ΩM. In course of integration, some precautions are needed against singularities of the function u on the set $T(\partial M)$. For this reason we will proceed as follows. Let $r : M \to \mathbf{R}$ be the distance to ∂M in the metric g. In some neighborhood of ∂M this function is smooth, and the boundary of the manifold $M_\rho = \{x \in M \mid r(x) \geq \rho\}$ is strictly convex for sufficiently small $\rho > 0$. The function u is smooth on ΩM_ρ, since $\Omega M_\rho \subset T^0 M \setminus T(\partial M)$. We multiply (4.3.4) by the volume form $d\Sigma = d\Sigma^{2n-1}$ and integrate it over ΩM_ρ. Transforming then the right-hand side of the so-obtained equality by the Gauss-Ostrogradskiĭ formulas (2.7.1) and (2.7.2), we obtain

$$\int\limits_{\Omega M_\rho} (|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u)\, d\Sigma = \int\limits_{\partial\Omega M_\rho} \langle \tilde{v} - v, \nu \rangle\, d\Sigma^{2n-2} - (n + 2m - 2)\int\limits_{\Omega M_\rho} \langle w, \xi \rangle\, d\Sigma. \quad (4.3.5)$$

The factor $n + 2m - 2$ is written before the last integral because the field $w(x, \xi)$ is homogeneous of degree $2m - 1$ in its second argument, as one can see from (3.5.6) and (4.1.4). Besides, (4.1.4) implies that $\langle w, \xi \rangle = (Hu)^2$ and, consequently, equality (4.3.5) takes the form

$$\int\limits_{\Omega M_\rho} \left[|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u + (n + 2m - 2)(Hu)^2 \right] d\Sigma = \int\limits_{\partial\Omega M_\rho} \langle \tilde{v} - v, \nu \rangle\, d\Sigma^{2n-2}. \quad (4.3.6)$$

Here $\nu = \nu_\rho(x)$ is the unit vector of the outer normal to the boundary of the manifold M_ρ.

We wish now to pass to the limit in (4.3.6) as $\rho \to 0$. To this end, we first note that both the sides of the last equality can be represented as integrals over domains independent of ρ. Indeed, since $\Omega M_\rho \subset \Omega M$, the domain of integration ΩM_ρ for the leftmost integral on (4.3.6) can be replaced with ΩM by multiplying simultaneously the integrand by the characteristic function $\chi_\rho(x)$ of the set M_ρ. The right-hand side of (4.3.6) can be transformed to an integral over $\partial\Omega M$ with the help of the diffeomorphism $\mu : \partial\Omega M \to \partial\Omega M_\rho$ defined by the equality $\mu(x, \xi) = (x', \xi')$, where a point x' is such that the geodesic $\gamma_{xx'}$, whose endpoints are x and x', has length ρ and intersects ∂M orthogonally at the x, and the vector ξ' is obtained by the parallel translation of the vector ξ along $\gamma_{xx'}$.

The integrands of (4.3.6) are smooth on $\Omega M \setminus \partial_0 \Omega M$ and, consequently, converge to their values almost everywhere on $\partial\Omega M$ as $\rho \to 0$. We also note that the first and third summands in the integrand on the left-hand side of (4.3.6) are nonnegative. Therefore, to apply the Lebesgue dominated convergence theorem, it remains to show that: 1) the second summand in the integrand on the left-hand side of (4.3.6) is summable over ΩM and 2) the absolute value of the integrand on the right-hand side of (4.3.6) is majorized by a function independent of ρ and summable over $\partial\Omega M$. We shall demonstrate more, namely, that the absolute values of the integrand on the right-hand side and of the second summand in the integrand on the left-hand side of (4.3.6) are bounded by some constant independent of ρ. Indeed, since these expressions are invariant, i.e., independent of the choice of coordinates, to prove our claim it suffices to show that these functions are bounded in the domain of some local coordinate system.

In a neighborhood of a point $x_0 \in \partial M$ we introduce a semigeodesic coordinate system similarly as in Lemma 3.2.3. Then $g_{in} = \delta_{in}$, $\nu^i = -\delta_n^i$. It follows from (4.1.3) and (4.3.2) that

$$\langle \tilde{v} - v, \nu \rangle = \xi^n \overset{h}{\nabla}^\alpha u \cdot \overset{v}{\nabla}_\alpha u - \xi^\alpha \overset{h}{\nabla}_\alpha u \cdot \overset{v}{\nabla}^n u - 2mu\, f_{n\, i_1 \ldots i_{m-1}} \xi^{i_1} \ldots \xi^{i_{m-1}}, \qquad (4.3.7)$$

In this formula (and in formula (4.3.9) below) the summation from 1 to $n-1$ over the index α is assumed. It is important that the right-hand side of (4.3.7) does not contain $\overset{h}{\nabla}_n u$. It follows from Lemma 3.2.3 and equality (3.5.1) that the derivatives $\overset{h}{\nabla}_\alpha u$ ($1 \leq \alpha \leq n - 1$) and $\overset{v}{\nabla}_i u$ ($1 \leq i \leq n$) are locally bounded.

Thus we have shown that passage to the limit is possible in (4.3.6) as $\rho \to 0$. Accomplishing it, we obtain the equality

$$\int\limits_{\Omega M} \left[|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u + (n + 2m - 2)(Hu)^2 \right] d\Sigma = \int\limits_{\partial\Omega M} (Lu - 2mu\, \langle j_\nu f, \xi^{m-1} \rangle)\, d\Sigma^{2n-2},$$

$$(4.3.8)$$

where L is the differential operator given in a semigeodesic coordinate system by the formula

$$Lu = \xi^n \overset{h}{\nabla}^\alpha u \cdot \overset{v}{\nabla}_\alpha u - \xi^\alpha \overset{h}{\nabla}_\alpha u \cdot \overset{v}{\nabla}^n u. \qquad (4.3.9)$$

Note that until now we did not use the restriction on sectional curvatures and boundary conditions (3.5.2)–(3.5.3); i.e., integral identity (4.3.8) is valid, for every solenoidal field f on an arbitrary CDRM, in which the function u is defined by formula (3.5.1).

In view of the boundary conditions (3.5.2)–(3.5.3), equality (4.3.8) can be written as:

$$\int\limits_{\Omega M} \left[|\overset{h}{\nabla} u|^2 - R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u + (n+2m-2)(Hu)^2 \right] d\Sigma = \int\limits_{\partial_+\Omega M} (L(If) - 2m(If)\langle j_\nu f, \xi^{m-1}\rangle)\, d\Sigma^{2n-2}.$$

(4.3.10)

If $(y^1, \ldots, y^{2n-2})$ is a local coordinate system on $\partial_+\Omega M$, then we see from (4.3.9) that Lu is a quadratic form in variables u, $\partial u/\partial y^i$ and $\partial u/\partial|\xi|$. According to homogeneity (3.5.6), $\partial u/\partial|\xi| = (m-1)u$ and, consequently, L is a quadratic first-order differential operator on the manifold $\partial_+\Omega M$. So the absolute value of the right-hand side of relation (4.3.10) is not greater than $C(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2)$ with some constant C independent of f. Consequently, (4.3.10) implies the inequality

$$\int\limits_{\Omega M} |\overset{h}{\nabla} u|^2 d\Sigma + (n+2m-2)\int\limits_{\Omega M} (Hu)^2\, d\Sigma \le \int\limits_{\Omega M} R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u\, d\Sigma + C(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2).$$

(4.3.11)

We introduce the semibasic covector fields y and z by the equalities

$$\overset{v}{\nabla}_i u = (m-1)\frac{u}{|\xi|^2}\xi_i + y_i,$$

(4.3.12)

$$\overset{h}{\nabla}_i u = \frac{Hu}{|\xi|^2}\xi_i + z_i.$$

(4.3.13)

Then y and z are orthogonal to ξ

$$\langle y, \xi \rangle = 0, \quad \langle z, \xi \rangle = 0.$$

The first of these equalities holds because $\langle \overset{v}{\nabla} u, \xi \rangle = (m-1)u$ as follows from homogeneity (3.5.6) of u. In particular,

$$|\overset{h}{\nabla} u|^2 = |z|^2 + \frac{(Hu)^2}{|\xi|^2}.$$

(4.3.14)

Using symmetries of the curvature tensor, we derive from (4.3.12)

$$R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u = R_{ijkl}\xi^i\xi^k y^j y^l.$$

(4.3.15)

With the help of (4.3.14) and (4.3.15), inequality (4.3.11) obtains the form

$$\int\limits_{\Omega M} |z|^2 d\Sigma + (n+2m-1)\int\limits_{\Omega M} (Hu)^2\, d\Sigma \le \int\limits_{\Omega M} R_{ijkl}\xi^i\xi^k y^j y^l\, d\Sigma + C(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2).$$ (4.3.16)

It turns out that the integral on the right-hand side of (4.3.16) can be estimated from above by the left-hand side of this inequality. To prove this fact, we first note that, for $(x, \xi) \in \Omega M$, the integrand on the right-hand side of (4.3.16) can be estimated as:

$$R_{ijkl}\xi^i\xi^k y^j y^l \le K^+(x, \xi)|y|^2,$$

(4.3.17)

where $K^+(x, \xi)$ is defined by formula (3.4.2).

In view of the boundary condition (3.5.3), the field y satisfies the conditions of Lemma 4.2.1. Applying this lemma, we obtain the estimate

$$\int\limits_{\Omega M} K^+(x, \xi)|y|^2 d\Sigma \le k^+ \int\limits_{\Omega M} |Hy|^2 d\Sigma,$$

(4.3.18)

where $k^+ = k^+(M, g)$ is given by equality (3.4.3).

We have to estimate $|Hy|^2$. Applying the operator H to equality (4.3.12) and using the commutation formula

$$\overset{v}{\nabla} H - H\overset{v}{\nabla} = \overset{h}{\nabla}$$

(4.3.19)

that follows from the definition of H, we obtain

$$Hy = \overset{v}{\nabla} Hu - \overset{h}{\nabla} u - (m-1)\frac{Hu}{|\xi|^2}\xi.$$

Consequently, for $|\xi| = 1$,

$$|Hy|^2 = |\overset{h}{\nabla} u|^2 + |\overset{v}{\nabla} Hu|^2 + (m-1)^2(Hu)^2 - 2(m-1)Hu\langle\overset{v}{\nabla} Hu, \xi\rangle + 2(m-1)Hu\langle\overset{h}{\nabla} u, \xi\rangle - 2\langle\overset{v}{\nabla} Hu, \overset{h}{\nabla} u\rangle.$$
$$(4.3.20)$$

Using (4.3.3), (4.3.14) and the equalities

$$\langle\overset{h}{\nabla} u, \xi\rangle = Hu, \quad \langle\overset{v}{\nabla} Hu, \xi\rangle = mHu,$$

we transform (4.3.20) to the form

$$|Hy|^2 = |z|^2 + |\overset{v}{\nabla} Hu|^2 + m(2-m)(Hu)^2 - \overset{h}{\nabla}_i \tilde{v}^i.$$

Integrating this equality and transforming the integral of the last term by the Gauss — Ostrogradskiĭ formula, we obtain

$$\int\limits_{\Omega M} |Hy|^2 d\Sigma = \int\limits_{\Omega M} |z|^2 d\Sigma + \int\limits_{\Omega M} |\overset{v}{\nabla} Hu|^2 d\Sigma + m(2-m)\int\limits_{\Omega M} (Hu)^2 d\Sigma - \int\limits_{\partial\Omega M} \langle\tilde{v}, \nu\rangle d\Sigma^{2n-2}. \qquad (4.3.21)$$

By the kinetic equation (3.5.4), $\overset{v}{\nabla} Hu = mF$, where F is defined by formula (4.2.17). Therefore the second integral on the right-hand side of (4.3.21) can be estimated with the help of Lemma 4.2.3 as follows:

$$\int\limits_{\Omega M} |\overset{v}{\nabla} Hu|^2 d\Sigma \le m(n+2m-2)\int\limits_{\Omega M} (Hu)^2 d\Sigma. \qquad (4.3.22)$$

(4.3.21) and (4.3.22) imply the inequality

$$\int\limits_{\Omega M} |Hy|^2 d\Sigma \le \int\limits_{\Omega M} |z|^2 d\Sigma + m(n+m)\int\limits_{\Omega M} (Hu)^2 d\Sigma - \int\limits_{\partial\Omega M} \langle\tilde{v}, \nu\rangle d\Sigma^{2n-2}.$$

Substituting the value (4.3.2) for $\tilde{v}$ and using the boundary conditions (3.5.2) and (3.5.3), we obtain

$$\int\limits_{\Omega M} |Hy|^2 d\Sigma \le \int\limits_{\Omega M} |z|^2 d\Sigma + m(n+m)\int\limits_{\Omega M} (Hu)^2 d\Sigma - 2m\int\limits_{\partial_+\Omega M} (If)\langle j_\nu f, \xi^{m-1}\rangle d\Sigma^{2n-2}.$$

Together with (4.3.17) and (4.3.18), the latter inequality gives

$$\int\limits_{\Omega M} R_{ijkl}\xi^i\xi^k y^j y^l \, d\Sigma \le k^+\left[\int\limits_{\Omega M} |z|^2 d\Sigma + m(n+m)\int\limits_{\Omega M} (Hu)^2 d\Sigma - 2m\int\limits_{\partial_+\Omega M} (If)\langle j_\nu f, \xi^{m-1}\rangle d\Sigma^{2n-2}\right].$$
$$(4.3.23)$$

We have already estimated the last integral on the right-hand side of (4.3.23):

$$\left|\int\limits_{\partial\Omega M} (If)\langle j_\nu f, \xi^{m-1}\rangle d\Sigma^{2n-2}\right| \le Cm\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0.$$

With the help of this estimate, (4.3.23) gives

$$\int\limits_{\Omega M} R_{ijkl}\xi^i\xi^k y^j y^l \, d\Sigma \le k^+\left[\int\limits_{\Omega M} |z|^2 d\Sigma + m(n+m)\int\limits_{\Omega M} (Hu)^2 d\Sigma + Cm\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0\right]. \qquad (4.3.24)$$

Estimating the right-hand integral of (4.3.16) with the help of (4.3.24), we arrive at the inequality

$$(1-k^+)\int\limits_{\Omega M} |z|^2 d\Sigma + [n+2m-1-m(n+m)k^+]\int\limits_{\Omega M} (Hu)^2 d\Sigma \le C\left(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2\right). \quad (4.3.25)$$

Under assumption (3.4.4) of Theorem 3.4.3, both coefficients at the integrals on (4.3.25) are positive and we obtain the estimate

$$\int_{\Omega M} (Hu)^2 d\Sigma \leq C_1(m\|If\|_0 \cdot \|j_\nu f|_{\partial M}\|_0 + \|If\|_1^2). \tag{4.3.26}$$

It remains to note that equation (3.5.4) implies the estimate

$$\|f\|_0^2 \leq C_2 \int_{\Omega M} (Hu)^2 d\Sigma.$$

From (4.3.26) with the help of the last estimate we obtain (3.4.8). Lemma 3.4.4 is proved as well as Theorem 3.4.3.

4.4 Deformation boundary rigidity of a Riemannian surface without focal points

Here we present an alternative approach to Problem 3.4.2 in the case of $n = \dim M = 2$ and $m = 2$. The approach is based on using isothermic coordinates.

A Riemannian manifold (M, g) *has no focal points* if, for every geodesic $\gamma : [a, b] \to M$ and every Jacobi field $Y(t)$ along γ satisfying the initial condition $Y(a) = 0$, the module $|Y(t)|$ is a strictly increasing function on $[a, b]$, i.e., $d|Y(t)|^2/dt > 0$ for $t \in [a, b]$.

The main result of the present section is the following

Theorem 4.4.1 *A two-dimensional Riemannian manifold (M, g) with strictly convex boundary and with no focal points is deformation boundary rigid, i.e., for a field $f \in C^\infty(S^2\tau'_M)$, the equality $If = 0$ implies existence of a covector field $v \in C^\infty(\tau'_M)$ such that $v|_{\partial M} = 0$ and $f = dv$.*

If a Riemannian manifold has no focal points, then it has no conjugate points. This implies that a manifold in Theorem 4.4.1 is simple and, in particular, is diffeomorphic to the disk D^2; compare with the remark after Theorem 3.4.3. It is known [46, 86] that there is a global system of isothermic coordinates on such a surface. This means that we can consider M as a closed bounded domain in the plain, $M \subset \mathbf{R}^2$, with a smooth boundary curve, and the metric g is of the form

$$ds^2 = e^{2\mu(x,y)}(dx^2 + dy^2) \tag{4.4.1}$$

with $\mu \in C^\infty(M)$. The Christoffel symbols of the metric are

$$\Gamma^1_{11} = \mu_x, \quad \Gamma^1_{12} = \mu_y, \quad \Gamma^1_{22} = -\mu_x,$$

$$\Gamma^2_{11} = -\mu_y, \quad \Gamma^2_{12} = \mu_x, \quad \Gamma^2_{22} = \mu_y.$$

We use the coordinates (x, y, θ) on the manifold ΩM of unit vectors, where θ is the angle from the horizontal direction to a vector. In other words,

$$\Omega M = \{(x, y; e^{-\mu}\cos\theta, e^{-\mu}\sin\theta) \mid (x, y) \in M, \theta \in \mathbf{R}\}.$$

The operator $H : C^\infty(\Omega M) \to C^\infty(\Omega M)$ of differentiation with respect to the geodesic flow looks as follows in these coordinates:

$$H = e^{-\mu}\left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + (-\mu_x\sin\theta + \mu_y\cos\theta)\frac{\partial}{\partial\theta}\right]. \tag{4.4.2}$$

Since the first factor does not matter, we introduce the new operator

$$L = e^\mu H = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + (-\mu_x\sin\theta + \mu_y\cos\theta)\frac{\partial}{\partial\theta} =$$

$$= \cos\theta\left(\frac{\partial}{\partial x} + \mu_y\frac{\partial}{\partial\theta}\right) + \sin\theta\left(\frac{\partial}{\partial y} - \mu_x\frac{\partial}{\partial\theta}\right). \tag{4.4.3}$$

Let $a = a(x, y, \theta)$ be an arbitrary smooth function (the modified function). The previous formula can be rewritten as follows:

$$L = \cos\theta\left(\frac{\partial}{\partial x} + (\mu_y - a\sin\theta)\frac{\partial}{\partial\theta}\right) + \sin\theta\left(\frac{\partial}{\partial y} + (-\mu_x + a\cos\theta)\frac{\partial}{\partial\theta}\right).$$

We introduce also the operator

$$L^\perp = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} - (\mu_x\cos\theta + \mu_y\sin\theta - a)\frac{\partial}{\partial\theta} =$$

$$= -\sin\theta\left(\frac{\partial}{\partial x} + (\mu_y - a\sin\theta)\frac{\partial}{\partial\theta}\right) + \cos\theta\left(\frac{\partial}{\partial y} + (-\mu_x + a\cos\theta)\frac{\partial}{\partial\theta}\right). \tag{4.4.4}$$

Lemma 4.4.2 *For every sufficiently smooth function $\varphi = \varphi(x, y, \theta)$, the following identity is valid*

$$2L^\perp\varphi \cdot \frac{\partial}{\partial\theta}(L\varphi) = [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]^2 + [\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]^2 +$$

$$+ \frac{\partial}{\partial x}\left[\varphi_y\varphi_\theta + (-\mu_x + a\cos\theta)\varphi_\theta^2\right] - \frac{\partial}{\partial y}\left[\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_\theta^2\right] +$$

$$+ \frac{\partial}{\partial\theta}\left[L^\perp\varphi \cdot L\varphi + (\mu_x - a\cos\theta)\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_y\varphi_\theta\right] -$$

$$- e^{2\mu}\left[H(e^{-\mu}a) + e^{-2\mu}a^2 + K\right]\varphi_\theta^2, \tag{4.4.5}$$

where K is the Gaussian curvature of metric (4.4.1).

This identity can be (and has been) obtained from the Pestov identity with modified horizontal derivative (see formula (6.1.28) below) by changing coordinates. However the changing is related to bulky calculations. The straightforward proof is easier.

Proof. First of all we remind that the Gaussian curvature of metric (4.4.1) is given by the expression $K = -e^{-2\mu}\Delta\mu$. Inserting the expression into (4.4.5) and introducing the notations

$$\alpha = \varphi_x + (\mu_y - a\sin\theta)\varphi_\theta, \qquad \beta = \varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta, \tag{4.4.6}$$

we have to prove the equality

$$2(-\alpha\sin\theta + \beta\cos\theta)\frac{\partial}{\partial\theta}(\alpha\cos\theta + \beta\sin\theta) = \alpha^2 + \beta^2 + \frac{\partial}{\partial x}(\beta\varphi_\theta) - \frac{\partial}{\partial y}(\alpha\varphi_\theta) +$$

$$+ \frac{\partial}{\partial\theta}\left[(-\alpha\sin\theta + \beta\cos\theta)(\alpha\cos\theta + \beta\sin\theta) + (\mu_x - a\cos\theta)\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_y\varphi_\theta\right] -$$

$$- \left[(a_x + \mu_y a_\theta - \mu_x a)\cos\theta + (a_y - \mu_x a_\theta - \mu_y a)\sin\theta + a^2 - \Delta\mu\right]\varphi_\theta^2.$$

It can be rewritten in the form

$$(-\alpha\sin\theta + \beta\cos\theta)\frac{\partial}{\partial\theta}(\alpha\cos\theta + \beta\sin\theta) - \frac{\partial}{\partial\theta}(-\alpha\sin\theta + \beta\cos\theta) \cdot (\alpha\cos\theta + \beta\sin\theta) =$$

$$= \alpha^2 + \beta^2 + \frac{\partial}{\partial x}(\beta\varphi_\theta) - \frac{\partial}{\partial y}(\alpha\varphi_\theta) + \frac{\partial}{\partial\theta}\left[(\mu_x - a\cos\theta)\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_y\varphi_\theta\right] -$$

$$- \left[(a_x + \mu_y a_\theta - \mu_x a)\cos\theta + (a_y - \mu_x a_\theta - \mu_y a)\sin\theta + a^2 - \Delta\mu\right]\varphi_\theta^2.$$

Implementing differentiations, we obtain

$$(-\alpha\sin\theta + \beta\cos\theta)(\alpha_\theta\cos\theta + \beta_\theta\sin\theta) - (-\alpha_\theta\sin\theta + \beta_\theta\cos\theta)(\alpha\cos\theta + \beta\sin\theta) =$$

$$= \beta\varphi_{x\theta} - \alpha\varphi_{y\theta} + (\beta_x - \alpha_y)\varphi_\theta + [(\mu_x - a\cos\theta)\varphi_x + (\mu_y - a\sin\theta)\varphi_y]\varphi_{\theta\theta} +$$

$$+ [(\mu_x - a\cos\theta)\varphi_{x\theta} + (\mu_y - a\sin\theta)\varphi_{y\theta} - (a_\theta\cos\theta - a\sin\theta)\varphi_x - (a_\theta\sin\theta + a\cos\theta)\varphi_y]\varphi_\theta -$$

$$- \left[(a_x + \mu_y a_\theta - \mu_x a)\cos\theta + (a_y - \mu_x a_\theta - \mu_y a)\sin\theta + a^2 - \Delta\mu\right]\varphi_\theta^2.$$

Opening parentheses on the left-hand side, we obtain

$$\alpha_\theta\beta - \beta_\theta\alpha = \beta\varphi_{x\theta} - \alpha\varphi_{y\theta} + [(\mu_x - a\cos\theta)\varphi_x + (\mu_y - a\sin\theta)\varphi_y]\varphi_{\theta\theta} + (\beta_x - \alpha_y)\varphi_\theta +$$

$$+ [(\mu_x - a\cos\theta)\varphi_{x\theta} + (\mu_y - a\sin\theta)\varphi_{y\theta} - (a_\theta\cos\theta - a\sin\theta)\varphi_x - (a_\theta\sin\theta + a\cos\theta)\varphi_y]\,\varphi_\theta -$$
$$- \left[(a_x + \mu_y a_\theta - \mu_x a)\cos\theta + (a_y - \mu_x a_\theta - \mu_y a)\sin\theta + a^2 - \Delta\mu\right]\varphi_\theta^2.$$

Substituting expressions (4.4.6) into the last equality, we obtain

$$[\varphi_{x\theta} + (\mu_y - a\sin\theta)\varphi_{\theta\theta} + (-a_\theta\sin\theta - a\cos\theta)\varphi_\theta]\,[\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta] -$$
$$- [\varphi_{y\theta} + (-\mu_x + a\cos\theta)\varphi_{\theta\theta} + (a_\theta\cos\theta - a\sin\theta)\varphi_\theta]\,[\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta] =$$
$$= [\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]\,\varphi_{x\theta} - [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]\,\varphi_{y\theta} +$$
$$+ [(\mu_x - a\cos\theta)\varphi_x + (\mu_y - a\sin\theta)\varphi_y]\,\varphi_{\theta\theta} +$$
$$+ [(-\mu_{xx} + a_x\cos\theta)\varphi_\theta - (\mu_{yy} - a_y\sin\theta)\varphi_\theta - (a_\theta\cos\theta - a\sin\theta)\varphi_x - (a_\theta\sin\theta + a\cos\theta)\varphi_y]\,\varphi_\theta -$$
$$- \left[(a_x + \mu_y a_\theta - \mu_x a)\cos\theta + (a_y - \mu_x a_\theta - \mu_y a)\sin\theta + a^2 - \Delta\mu\right]\varphi_\theta^2.$$

After canceling some terms on the right-hand side, this equality takes the form

$$[\varphi_{x\theta} + (\mu_y - a\sin\theta)\varphi_{\theta\theta} + (-a_\theta\sin\theta - a\cos\theta)\varphi_\theta]\,[\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta] -$$
$$- [\varphi_{y\theta} + (-\mu_x + a\cos\theta)\varphi_{\theta\theta} + (a_\theta\cos\theta - a\sin\theta)\varphi_\theta]\,[\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta] =$$
$$= [\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]\,\varphi_{x\theta} - [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]\,\varphi_{y\theta} +$$
$$+ [(\mu_x - a\cos\theta)\varphi_x + (\mu_y - a\sin\theta)\varphi_y]\,\varphi_{\theta\theta} +$$
$$+ [(-a_\theta\cos\theta + a\sin\theta)\varphi_x + (-a_\theta\sin\theta - a\cos\theta)\varphi_y]\,\varphi_\theta -$$
$$- \left[(\mu_y a_\theta - \mu_x a)\cos\theta + (-\mu_x a_\theta - \mu_y a)\sin\theta + a^2\right]\varphi_\theta^2.$$

Grouping together the terms containing the same second-order derivatives, we transform the equality to the following one:

$$[\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]\,\varphi_{x\theta} - [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]\,\varphi_{y\theta} +$$
$$+ [(\mu_y - a\sin\theta)\,(\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta) - (-\mu_x + a\cos\theta)\,(\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta)]\,\varphi_{\theta\theta} +$$
$$+ [(-a_\theta\sin\theta - a\cos\theta)\varphi_y - (a_\theta\cos\theta - a\sin\theta)\varphi_x]\,\varphi_\theta +$$
$$+ [(-a_\theta\sin\theta - a\cos\theta)(-\mu_x + a\cos\theta) - (a_\theta\cos\theta - a\sin\theta)(\mu_y - a\sin\theta)]\,\varphi_\theta^2 =$$
$$= [\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]\,\varphi_{x\theta} - [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]\,\varphi_{y\theta} +$$
$$+ [(\mu_x - a\cos\theta)\varphi_x + (\mu_y - a\sin\theta)\varphi_y]\,\varphi_{\theta\theta} +$$
$$+ [(-a_\theta\cos\theta + a\sin\theta)\varphi_x + (-a_\theta\sin\theta - a\cos\theta)\varphi_y]\,\varphi_\theta -$$
$$- \left[(\mu_y a_\theta - \mu_x a)\cos\theta + (-\mu_x a_\theta - \mu_y a)\sin\theta + a^2\right]\varphi_\theta^2.$$

After canceling some terms, this equality takes the form

$$[(-a_\theta\sin\theta - a\cos\theta)(-\mu_x + a\cos\theta) - (a_\theta\cos\theta - a\sin\theta)(\mu_y - a\sin\theta)]\,\varphi_\theta^2 =$$
$$= - \left[(\mu_y a_\theta - \mu_x a)\cos\theta + (-\mu_x a_\theta - \mu_y a)\sin\theta + a^2\right]\varphi_\theta^2.$$

The latter equality evidently holds. The lemma is proved.

Lemma 4.4.3 *If a function* $u \in C^4(\Omega M)$ *satisfies the equation*

$$Lu = \frac{1}{2}f_0 + f_1\cos 2\theta + f_2\sin 2\theta \tag{4.4.7}$$

with $f_i = f_i(x, y)$, *then the function*

$$\varphi = u_{\theta\theta} + u$$

satisfies the equality

$$-4(L^\perp\varphi - \frac{1}{2}a\varphi_\theta)^2 = [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]^2 + [\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]^2 +$$

$$+ \frac{\partial}{\partial x}\left[\varphi_y\varphi_\theta + (-\mu_x + a\cos\theta)\varphi_\theta^2\right] - \frac{\partial}{\partial y}\left[\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_\theta^2\right] +$$

$$+ \frac{\partial}{\partial\theta}\left[L^\perp\varphi \cdot L\varphi + (\mu_x - a\cos\theta)\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_y\varphi_\theta\right] -$$

$$- e^{2\mu}\left[H(e^{-\mu}a) + 2e^{-2\mu}a^2 + K\right]\varphi_\theta^2. \tag{4.4.8}$$

Proof. The definitions of L and $L^\perp$ imply the commutation formulas

$$\frac{\partial}{\partial \theta}L - L\frac{\partial}{\partial \theta} = L^\perp - a\frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial \theta}L^\perp - L^\perp\frac{\partial}{\partial \theta} = -L + a_\theta\frac{\partial}{\partial \theta}. \tag{4.4.9}$$

Differentiating equation (4.4.7) with respect to θ and applying the first of formulas (4.4.9), we obtain

$$Lu_\theta + L^\perp u - au_\theta = 2(-f_1 \sin 2\theta + f_2 \cos 2\theta). \tag{4.4.10}$$

Differentiating (4.4.10) with respect to θ and applying (4.4.9) again, we obtain

$$Lu_{\theta\theta} - 2au_{\theta\theta} + 2L^\perp u_\theta - Lu = -4(f_1 \cos 2\theta + f_2 \sin 2\theta). \tag{4.4.11}$$

Multiplying equation (4.4.7) by 2 and adding the result to (4.4.11), we get the equality

$$L(u_{\theta\theta} + u) - 2au_{\theta\theta} + 2L^\perp u_\theta = f_0 - 2(f_1 \cos 2\theta + f_2 \sin 2\theta)$$

that can be rewritten in the form

$$L\varphi = -2L^\perp u_\theta + 2au_{\theta\theta} + F \tag{4.4.12}$$

with

$$F = f_0 - 2(f_1 \cos 2\theta + f_2 \sin 2\theta). \tag{4.4.13}$$

By (4.4.12),

$$\frac{\partial}{\partial \theta}(L\varphi) = -2\frac{\partial}{\partial \theta}L^\perp u_\theta + 2au_{\theta\theta\theta} + 2a_\theta u_{\theta\theta} + F_\theta.$$

Using the second of commutation formulas (4.4.9), we rewrite the latter equality in the form

$$\frac{\partial}{\partial \theta}(L\varphi) = -2L^\perp u_{\theta\theta} + 2Lu_\theta + 2au_{\theta\theta\theta} + F_\theta. \tag{4.4.14}$$

By (4.4.10),

$$Lu_\theta = -L^\perp u + au_\theta + 2(-f_1 \sin 2\theta + f_2 \cos 2\theta).$$

Inserting this expression into (4.4.14), we obtain

$$\frac{\partial}{\partial \theta}(L\varphi) = -2L^\perp u_{\theta\theta} - 2L^\perp u + 2au_\theta + 2au_{\theta\theta\theta} + 4(-f_1 \sin 2\theta + f_2 \cos 2\theta) + F_\theta.$$

By (4.4.13), the sum of two last terms on the right-hand side of the latter equality is equal to zero, and we arrive at the relation

$$\frac{\partial}{\partial \theta}(L\varphi) = -2L^\perp \varphi + 2a\varphi_\theta. \tag{4.4.15}$$

We write down the Pestov identity (4.4.5) for the function φ. By (4.4.15),

$$2L^\perp \varphi \cdot \frac{\partial}{\partial \theta}(L\varphi) = -4(L^\perp \varphi)^2 + 4aL^\perp \varphi \cdot \varphi_\theta = -4\left(L^\perp \varphi - \frac{1}{2}a\varphi_\theta\right)^2 + a^2\varphi_\theta^2. \tag{4.4.16}$$

Inserting the expression (4.4.16) into the left-hand side of (4.4.5), we arrive at (4.4.8). The lemma is proved.

Lemma 4.4.4 *Under hypotheses of Theorem 4.4.1, there exists a function $b \in C^\infty(\Omega M)$ satisfying the inequality*

$$Hb + 2b^2 + K \le 0, \tag{4.4.17}$$

where K is the Gaussian curvature.

Proof. Let us fix a unit speed geodesic $\gamma : [0, l] \to M$ with endpoints on the boundary, $\gamma(0), \gamma(l) \in \partial M$. Let $x = x(t), y = y(t), \theta = \theta(t)$ be the parametric equations of γ in the isothermic coordinates. We will first prove that inequality (4.4.17) has a solution on γ. Putting $x = x(t), y = y(t), \theta = \theta(t)$ in (4.4.17), we arrive at the inequality

$$\dot{b} + 2b^2 + K \le 0. \tag{4.4.18}$$

By the change $b = a/2$, the inequality is transformed to the following one:

$$\dot{a} + a^2 + 2K \le 0. \tag{4.4.19}$$

Since the geodesic γ has no focal points, the Jacobi equation

$$\ddot{y} + Ky = 0 \tag{4.4.20}$$

has a positive solution y_1^γ with the positive derivative $\dot{y}_1^\gamma$ on $[0, l]$. Consequently, the function $a_1 = \dot{y}_1^\gamma / y_1^\gamma$ is positive and satisfies the Riccati equation

$$\dot{a}_1 + a_1^2 + K = 0. \tag{4.4.21}$$

Applying the same argument to the geodesic γ with the reversed orientation, we obtain a positive solution y_2^γ to the Jacobi equation (4.4.20) with the negative derivative $\dot{y}_2^\gamma$ on $[0, l]$. Consequently, the function $a_2 = \dot{y}_2^\gamma / y_2^\gamma$ is negative and satisfies the Riccati equation

$$\dot{a}_2 + a_2^2 + K = 0. \tag{4.4.22}$$

Summing (4.4.21) and (4.4.22), we obtain for the function $a = a_1 + a_2$

$$\dot{a} + a^2 + 2K = 2a_1 a_2.$$

Since the functions a_1 and a_2 have different signs, we see that inequality (4.4.19) is satisfied by a.

We represent ΩM as the union of disjoint curves, the orbits of the geodesic flow, which are geodesics considered as curves in ΩM. We have proved that inequality (4.4.17) has a solution on every such curve. We have now to choose these solutions in such a way that their union gives us a function that is smooth on the whole of ΩM. To this end we observe that the above-discussed construction of the function b has only the following arbitrariness: the choice of the initial values $y_i^\gamma(0), \dot{y}_i^\gamma(0)$ $(i = 1, 2)$ of the solutions to the Jacobi equation. The family of oriented geodesics γ can be parameterized by points of the product $\Omega^1 \times \Omega^1$ of two circles. Choosing smooth on $\Omega^1 \times \Omega^1$ functions $y_i^\gamma(0), \dot{y}_i^\gamma(0)$, we obtain the smooth on ΩM solution b to inequality (4.4.17). The lemma is proved.

Proof of Theorem 4.4.1. Let a tensor field $f \in C^\infty(S^2 \tau_M')$ lie in the kernel of the ray transform, $If = 0$. We define the function $u \in C(\Omega M)$ by formula (3.5.1) with $m = 2$. This function satisfies the kinetic equation

$$Hu = f_{ij}(x)\xi^i \xi^j \tag{4.4.23}$$

and the homogeneous boundary condition

$$u(x, \xi)|_{x \in \partial M} = 0. \tag{4.4.24}$$

The function $u(x, \xi)$ depends smoothly on $(x, \xi) \in \Omega M$ except of the points of the set $\Omega(\partial M)$ where some derivatives of u can be infinite. Consequently, some of the integrals considered below are improper and we have to verify their convergence. The verification is performed in the same way as in Section 4.3, since the singularities of u are due only to the singularities of the low integration limit in (3.5.1). In order to simplify the presentation, we will not pay attention to these singularities in what follows.

Being written in the isothermic coordinates, the kinetic equation (4.4.23) has the form (4.4.7) with

$$f_0 = \frac{1}{2}e^{-\mu}(f_{11} + f_{22}), \quad f_1 = \frac{1}{2}e^{-\mu}(f_{11} - f_{22}), \quad f_2 = e^{-\mu}f_{12}.$$

Put $a = e^\mu b$, where b is the function constructed in Lemma 4.4.4. We write down the equality (4.4.8) for the function $\varphi = u_{\theta\theta} + u$ and integrate it over ΩM. The integrals of divergent terms are equal to zero because the functions $\varphi(x, y, \theta)$ and $a(x, y, \theta)$ are 2π-periodical in θ, and the function $\varphi(x, y, \theta)$ vanishes for $(x, y) \in \partial M$ as follows from (4.4.24). We thus obtain after integration

$$\int_0^{2\pi}\int_M \left[(\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta)^2 + (\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta)^2 + 4(L^\perp\varphi - \frac{1}{2}a\varphi_\theta)^2 \right] dxdyd\theta =$$

$$= \int_0^{2\pi}\int_M e^{2\mu}\left(Hb + 2b^2 + K\right)\varphi_\theta^2\, dxdyd\theta. \tag{4.4.25}$$

By Lemma 4.4.4, the right-hand side of (4.4.25) is nonpositive. Since the integrand on the left-hand side is nonnegative, this implies that

$$\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta = 0, \quad \varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta = 0.$$

In particular,

$$L\varphi = \cos\theta \left[\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta\right] + \sin\theta \left[\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta\right] = 0.$$

The latter equation, together with the homogeneous boundary condition

$$\varphi|_{\partial\Omega M} = 0,$$

implies that

$$\varphi = u_{\theta\theta} + u = 0.$$

This means that the function u can be represented in the form

$$u(x, y, \theta) = c(x, y)\cos\theta + d(x, y)\sin\theta. \tag{4.4.26}$$

Substituting the expression (4.4.26) for u into the kinetic equation (4.4.7), we see that

$$f_{11} = e^\mu(c_x + \mu_y d), \quad f_{12} = \frac{1}{2}e^\mu(c_y + d_x - \mu_y c - \mu_x d), \quad f_{22} = e^\mu(d_y + \mu_x c).$$

This equalities are equivalent to the relation $f = dv$, where v is the covector field with the coordinates $v_1 = e^\mu c$ and $v_2 = e^\mu d$. Since v vanishes on ∂M by (4.4.24), the field f is potential. The theorem is proved.

4.5   Some remarks

The general scheme of the method used in the proof of Theorem 3.4.3 is known in mathematical physics for a long time under the name of the method of energy estimates or the method of quadratic integrals. At first, for classical equations, the main relations of the method had a physical sense of energy integrals. While being extended later to wider classes of equations, the method was treated in a more formal manner [27]. Roughly speaking, the principal idea of the method can be explained as follows: given a differential operator D, we try to find another differential operator L such that the product $LuDu$ can be decomposed into the sum of two summands in such a way that the first summand is presented in divergence form and the second one is a positive-definite quadratic form in the higher derivatives of the function u. The Pestov identity (4.1.2) is an example of such decomposition. The first summand on the right-hand side of (4.1.2) is a positive-definite quadratic form in derivatives $\partial u/\partial x^i$, the second and third summands are of divergence form while the last summand is considered, from the viewpoint of the method, as an undesirable term.

In integral geometry the method was at first applied by R. G. Mukhometov [55, 56, 57] to a two-dimensional problem. Thereafter this approach to integral geometry problems was developed by R. G. Mukhometov himself [58, 59, 60, 61] as well as others [8, 10, 14, 62, 69]. In this series, some papers due to A. Kh. Amirov [4, 5, 6, 7] can be distinguished where some new ideas have arisen.

The first and foremost difference between Mukhometov's approach and Amirov's one, which determines other distinctions, is the choice of the coordinates on the manifold ΩM. A. Kh. Amirov uses the same coordinates $(x, \xi) = (x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ as we have used in these lectures, while R. G. Mukhometov uses the coordinates $(x^1, \ldots, x^n, z^1, \ldots, z^{n-1})$, where $z \in \partial M$ is a point at which the geodesic $\gamma_{x,\xi}$ meets the boundary. Each of these coordinate systems has its own merits and demerits. For instance, being written in the coordinates (x, z), the kinetic equation is of a more simple structure (dose not contain the derivatives with respect to z). But at the same time, if the right-hand side of the equation depends on ξ in some way (for instance, in these lectures we are interested in the polynomial dependence on ξ), then in the coordinates (x, z) the dependence obtains very complicated character. On the other hand, using the coordinates (x, ξ), there is no problem with the right-hand side. But at the same time, since the equation contains the derivatives with respect to ξ, to apply the method successfully, we have to impose some assumptions, on the coefficients of the equation, which require positive definiteness for a quadratic form.

In [66] L. N. Pestov and V. A. Sharafutdinov implemented the method in covariant terms and demonstrated that, in this approach, the mentioned assumptions on the coefficients of the equation turn into the requirement of nonpositivity for the sectional curvature. Of course, the last requirement is more sensible from a geometrical standpoint.

In [73] the author used the coordinates (x, η) that differ from (x, ξ) as well as from (x, z). Namely η is the vector tangent to the geodesic $\gamma_{x,\xi}$ at the point z. By means of these coordinates, the claim

of Theorem 3.4.3 was obtained for a domain $M \subset \mathbf{R}^n$ and for metrics C^2-close to that of Euclidean space. The system (x, η) has the same advantage as (x, z), i.e., the kinetic equation does not contain the derivatives with respect to η. At the same time, if a metric is close to that of Euclidean space, the coordinates (x, η) and (x, ξ) are close. The last circumstance makes application of the method a full success.

Finally, in [75] the author noticed that the Poincaré inequality allows one to obtain estimate (4.3.24) that leads to Theorem 3.4.3 which unites and essentially strengthens the results of [66] and [73]. Note that this observation is of a rather general nature, i.e., it can be applied to other kinds of the kinetic equation, as we will see in the next lectures. On the other hand, this observation allows one to extend essentially the scope of the method, since making it possible to replace the assumptions of positive definiteness of a quadratic form by conditions of the type "of a slightly perturbed system". The last conditions often turns out to be more acceptable for a physical interpretation.

In [80] the author considered Problem 3.4.2 for spherically symmetric metrics on the ball $\{x \in \mathbf{R}^n \mid |x| \leq R\}$. For such a metric condition (3.4.4) can fail, and the techniques of the current lecture does not work. Nevertheless some result similar to Theorem 3.4.3 is proved with the help of Fourier series techniques.

Theorem 4.4.1 belongs to the author and G. Uhlmann [81]. It is interesting to compare theorems 3.4.3 and 4.4.1. In the case of $n = m = 2$, hypothesis (3.4.4) of Theorem 3.4.3 looks as follows:

$$k^+(M, g) = \sup_{\gamma} \int_0^l tK^+(\gamma(t))dt < \frac{5}{8}, \tag{4.5.1}$$

where the supremum is taken over all unit speed geodesics $\gamma : [0, l] \to M$, and $K^+ = \max\{K, 0\}$, K being the Gaussian curvature. It is easy to see that (4.5.1) does not follow from absence of focal points. For instance, let M be a convex domain in the unit sphere Ω^2. The domain has no focal points if $\operatorname{diam} M < \pi/2$, and satisfies (4.5.1) if $\operatorname{diam} M < \sqrt{5}/2$. Therefore Theorem 3.4.3 is not stronger than Theorem 4.4.1. On the other hand, the author does not know the answer to the question: does inequality (4.5.1) imply absence of focal points?

Lecture 5
Local boundary rigidity

Here we consider the local version of the nonlinear Problem 3.1.1. The term "local" means that the metrics g and g' participating in the problem are assumed to be close one to other. We solve the problem under the same assumption on curvature as in Theorem 3.4.3 on the corresponding linear problem.

In this lecture our presentation mostly follows the paper [20]. See also [84].

5.1 Statement of the result

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Given a natural number k and a real number α, $0 < \alpha < 1$, we denote by $\mathrm{Diff}_0^{k,\alpha}(M)$ the set of all diffeomorphisms of M onto itself that are the identity on the boundary and are given by functions of class $C_{loc}^{k,\alpha}$ in local coordinates of M. We endow $\mathrm{Diff}_0^{k,\alpha}(M)$ with the $C^{k,\alpha}$-topology, defining some $C^{k,\alpha}$-norm by means of a finite atlas and a subordinate partition of unity. The resultant topology is clearly independent of the choice of the norm.

We let $C^{k,\alpha}(S^2\tau'_M)$ stand for the space of $C^{k,\alpha}$-smooth covariant symmetric tensor fields of degree 2 on M. We endow $C^{k,\alpha}(S^2\tau'_M)$ with the natural $C^{k,\alpha}$-topology. Then $C^{k,\alpha}(S^2\tau'_M)$ becomes a topological Banach space, i.e., a topological vector space whose topology can be defined by some norm making it a Banach space.

Now, we are in a position to formulate the main result of this lecture.

Theorem 5.1.1 *Let an $n$dimensional CDRM (M, g) satisfy the condition*

$$k^+(M, g) < (n+3)/(2n+4), \tag{5.1.2}$$

where $k^+(M, g)$ is defined by (3.4.3). Then there is a neighborhood $W \subset C^{3,\alpha}(S^2\tau'_M)$ of g, with any $0 < \alpha < 1$, such that if a metric $g' \in W$ has the same boundary distance-function as g, $\Gamma_g = \Gamma_{g'}$, then there exists a diffeomorphism $\varphi : M \to M$ in $\mathrm{Diff}_0^{3,\alpha}(M)$ such that $g' = \varphi^*g$; moreover, φ tends to the identity as g' tends to g (both in $C^{3,\alpha}$-topology).

Remark. Condition (5.1.2) implies the simplicity of (M, g), cf. with the remark after Theorem 3.4.3. Inequality (5.1.2) holds for instance when M is nonpositively curved or is a sufficiently small convex piece of an arbitrary Riemannian manifold.

In the next few paragraphs we explain how the current lecture is organized. Each section treats a different aspect of the problem and many sections work more generally than when $\Gamma_g = \Gamma_{g'}$. For example, Sections 5.2 and 5.5 deal with any two sufficiently close metrics while Section 5.4 deals with a metric g' near a simple metric g such that $\Gamma_{g^1}(x, y) \geq \Gamma_g(x, y)$ for all x, y in the boundary.

In Section 5.2 we "shift" any tensor g' which is sufficiently close to a given metric g to a solenoidal one with respect to g. That is, we find a diffeomorphism $\varphi \in \mathrm{Diff}_0^{3,\alpha}(M)$ such that the pull-back $g^1 = \varphi^*g'$ of g' is a solenoidal tensor field with respect to g (i.e., the g-divergence of g^1 is 0).

In Section 5.3, we prove that the volume of a simple Riemannian manifold is determined by the boundary distance function.

In Section 5.4, we show that if g^1 is sufficiently close to a simple metric g and if, for all pairs x and y on the boundary, $\Gamma_{g^1}(x, y) \geq \Gamma_g(x, y)$ then the ray transform If of the tensor field $f = g^1 - g$ is nonnegative. Also using Santaló's formula we see that $\lambda \equiv (g, f)_{L_2(S^2\tau'_M)} \geq 0$.

In Section 5.5, we consider the volume of metrics g^1 which are sufficiently close to a given metric g. We show that if $Vol(g^1) \leq Vol(g)$ then the tensor field $f = g^1 - g$ satisfies $\lambda \leq \frac{2}{3}\|f\|^2_{L_2(S^2\tau'_M)}$.

In Section 5.6, we consider metrics g^1 close to a given dissipative metric g which induce the same Riemannian metric on the boundary as the one induced by g. The ray transform of $f = g^1 - g$ satisfies a number of useful properties that are exploited in section 5.7.

In Section 5.7, we complete the proof the main theorem with the help of Pestov's identity. In fact we show

Lemma 5.1.2 *For any metric g satisfying the assumptions of Theorem 5.1.1 there is a neighborhood $W \subset C^{3,\alpha}(S^2\tau'_M)$ of g, with any $0 < \alpha < 1$, such that if a metric $g^1 \in W$ induces the same Riemannian metric on the boundary as g and $\Gamma_{g^1}(x, y) \geq \Gamma_g(x, y)$ for all boundary points x and y then $Vol(g^1) \geq Vol(g)$ with equality if and only if g^1 is isometric to g.*

Theorem 5.1.1 follows directly from this since if g^1 and g have the same boundary distance function then they induce the same Riemannian metric on the boundary and they have the same volume.

Lemma 5.1.2 may be of some independent interest since little is understood about how inequalities between the boundary distance functions might relate the volumes of Riemannian manifolds with boundary, (see for example Gromov's notion of the filling volume [34]). The corresponding statement for the compact without boundary case would be: If g is a metric on a compact negatively curved manifold M and g' is a metric sufficiently close to g and such that the g'-length of each free homotopy class is $\geq$ the g-length then $Vol(g') \geq Vol(g)$ with equality holding if and only if g' is isometric to g. This statement remains an open question, but our results lend it support.

5.2   Shift of a tensor field to solenoidal one

Theorem 5.2.1 *Let (M, g) be a compact Riemannian manifold with convex boundary and let $k \geq 2$ be an integer and $0 < \alpha < 1$ a real. Then for every neighborhood $U \subset Diff_0^{k,\alpha}(M)$ of the identity there is a neighborhood $W \subset C^{k,\alpha}(S^2\tau'_M)$ of the metric tensor g such that for every metric $g' \in W$ there exists a diffeomorphism $\varphi \in U$ for which the tensor field φ^*g' is solenoidal; i.e., $\delta(\varphi^*g') = 0$, where δ is the divergence in the metric g.*

Remark. The assumption that the boundary is convex slightly simplifies the proof of the theorem but is not essential for its validity. A similar theorem holds for a closed (M, g) under the assumption that there exists a dense geodesic in ΩM.

The proof consists in applying a Banach space version of the implicit function theorem. To this end, we first of all must realize some neighborhood of the identity in $\mathrm{Diff}_0^{k,\alpha}(M)$ as an open set in a Banach space.

Denote by $C_0^{k,\alpha}(\tau_M)$ the topological Banach space of vector fields of class $C^{k,\alpha}$ on M which vanish on ∂M. Let Ω be the open neighborhood of the zero in $C_0^{k,\alpha}(\tau_M)$ $(k \geq 1)$ which consists of the vector fields v satisfying the inequality $|\nabla v| < 1$. This inequality and the boundary condition $v|_{\partial M} = 0$ imply that $|v(x)| < \mathrm{dist}\,(x, \partial M)$ for all $x \in M$. Therefore, the mapping

$$e_v : M \to M, \quad e_v(x) = \exp_x v(x) \tag{5.2.1}$$

is well-defined for all $v \in \Omega$. It is easy to check that there is some smaller neighborhood $\Omega' \subset \Omega$ of zero in $C_0^{k,\alpha}(\tau_M)$ such that $e_v \in \mathrm{Diff}_0^{k,\alpha}(M)$ for $v \in \Omega'$. The mapping

$$\Omega' \to \mathrm{Diff}_0^{k,\alpha}(M), \quad v \mapsto e_v \tag{5.2.2}$$

is continuous. The inverse of (5.2.2), $\varphi \mapsto v_\varphi$, is defined for $\varphi \in \mathrm{Diff}_0^{k,\alpha}(M)$ sufficiently close to the identity as follows: $v_\varphi(x) = \dot{\gamma}(0)$, where $\gamma : [0, 1] \to M$ is the geodesic such that $\gamma(0) = x$ and $\gamma(1) = \varphi(x)$; the existence of this geodesic is guaranteed by the convexity of the boundary. We thus establish that (5.2.2) is a homeomorphism of the neighborhood Ω' of the zero in the Banach space $C_0^{k,\alpha}(\tau_M)$ onto some neighborhood of the identity in the space $\mathrm{Diff}_0^{k,\alpha}(M)$. Therefore, the theorem will be proven once we prove the following assertion.

Lemma 5.2.2 *Under the conditions of the theorem, let $\Omega \subset C_0^{k,\alpha}(\tau_M)$ be a neighborhood of zero such that the mapping (5.2.1) is defined for all $v \in \Omega$. Then there exists a neighborhood $G \subset C_0^{k,\alpha}(S^2\tau'_M)$ of zero and a continuous mapping $\beta : G \to \Omega$ such that $\beta(0) = 0$ and the tensor field $(e_{\beta(f)})^*(g + f)$ is solenoidal (in the metric g) for all $f \in G$.*

Proof. Consider the mapping

$$F : \Omega \times C^{k,\alpha}(S^2\tau'_M) \to C^{k-2,\alpha}(\tau'_M) \tag{5.2.3}$$

defined by

$$F(v, f) = \delta(e_v^*(g + f)), \qquad v \in \Omega \subset C_0^{k,\alpha}(\tau_M), \ f \in C^{k,\alpha}(S^2\tau'_M). \tag{5.2.4}$$

We need to show that F is continuous and has continuous partial derivatives F'_v and F'_f. To this end, represent F as the composition

$$F(v, f) = \delta R(v, g + f), \tag{5.2.5}$$

where $\delta : C^{k-1,\alpha}(S^2\tau'_M) \to C^{k-2,\alpha}(\tau'_M)$ is the divergence in the metric g and the mapping

$$R : \Omega \times C^{k,\alpha}(S^2\tau'_M) \to C^{k-1,\alpha}(S^2\tau'_M) \tag{5.2.6}$$

is defined by

$$R(v, f) = e_v^* f. \tag{5.2.7}$$

Since δ is a first order linear differential operator, we have

$$F'_v(v, f) = \delta R'_v(v, g + f), \quad F'_f(v, f) = \delta R'_f(v, g + f). \tag{5.2.8}$$

Hence, the matter is reduced to verifying the continuity of the function R and of its derivatives R'_v and R'_f.

Let $(x^1, \ldots, x^n)$ be a local coordinate system on M with domain $U \subset M$. For $x \in U$ and a sufficiently small vector $\xi \in T_x M$, the point $\exp_x \xi$ belongs to U as well; we denote the coordinates of this point by $(E^1(x, \xi), \ldots, E^n(x, \xi))$. According to (5.2.1), the point $e_v(x)$ has coordinates $(e_v^1(x), \ldots, e_v^n(x))$ with

$$e_v^i(x) = E^i(x, v(x)). \tag{5.2.9}$$

Now, (5.2.7) is rewritten in coordinates as

$$(R(v, f))_{ij} = \frac{\partial e_v^p}{\partial x^i} \frac{\partial e_v^q}{\partial x^j} f_{pq} \circ e_v. \tag{5.2.10}$$

For every vector field $v \in C_0^{k,\alpha}(\tau_M)$, the function $e_v^p(x)$ is of class $C^{k,\alpha}$. The fact that the right-hand side of (5.2.10) lies in the space $C^{k-1,\alpha}(S^2\tau'_M)$ and that it has continuous dependence on (v, f) follow from the two facts:

(a) if $\varphi, \psi \in C^{k,\alpha}$ then the product $\varphi\psi$ also belongs to $C^{k,\alpha}$ and the mapping $C^{k,\alpha} \times C^{k,\alpha} \to C^{k,\alpha}$, $(\varphi, \psi) \mapsto \varphi\psi$ is continuous;

(b) if $\varphi, \psi \in C^{k,\alpha}$ $(k \geq 1)$ and the composition $\varphi \circ \psi$ is defined then $\varphi \circ \psi \in C^{k,\alpha}$ and the mapping $C^{k,\alpha} \times C^{k,\alpha} \to C^{k,\alpha}$, $(\varphi, \psi) \mapsto \varphi \circ \psi$ is continuous.

Since the mapping (5.2.6) is linear in f, the partial derivative R'_f is given by the expression $R'_f(v, f)\widetilde{f} = e_v^* \widetilde{f}$ and its continuity ensues from the same arguments as for R.

Differentiating (5.2.10) with respect to v, we find (using $f_{pq} = f_{qp}$) the partial derivative R'_v:

$$(R'_v(v, f)\widetilde{v})_{ij} = \left(\frac{\partial e_v^p}{\partial x^i} \frac{\partial (De_v^q)}{\partial x^j} + \frac{\partial e_v^p}{\partial x^j} \frac{\partial (De_v^q)}{\partial x^i} \right) f_{pq} \circ e_v + (De_v^r) \frac{\partial e_v^p}{\partial x^i} \frac{\partial e_v^q}{\partial x^j} \frac{\partial f_{pq}}{\partial x^r} \circ e_v, \tag{5.2.11}$$

where De_v^i is the variation of the function e_v^i which by (5.2.9) is given by the expression

$$De_v^i = \frac{\partial E^i}{\partial \xi^r}(x, v(x))\widetilde{v}^r(x). \tag{5.2.12}$$

Using (5.2.11) and (5.2.12), the continuity of R'_v follows from the same arguments as above.

We now compute $F'_v(0, 0)$. Setting $v = 0$, $f = g$ in (5.2.11), (5.2.12) and using the relations

$$e_v^i|_{v=0} = x^i, \quad \frac{\partial E^i}{\partial \xi^r}(x, 0) = \delta_r^i,$$

we find

$$(R'_v(0, g)\widetilde{v})_{ij} = g_{ip}\frac{\partial \widetilde{v}^p}{\partial x^j} + g_{jp}\frac{\partial \widetilde{v}^p}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^p}\widetilde{v}^p.$$

Rewriting the partial derivatives $\partial \widetilde{v}^p / \partial x^i$ in terms of the covariant derivatives $\nabla_i \widetilde{v}^p = \partial \widetilde{v}^p / \partial x^i + \Gamma_{iq}^p \widetilde{v}^q$, we arrive at the equality

$$(R_v'(0,g)\widetilde{v})_{ij} = \nabla_i \widetilde{v}_j + \nabla_j \widetilde{v}_i = 2(d\widetilde{v})_{ij},$$

where $\widetilde{v}_i = g_{ij}\widetilde{v}^j$ and $d = \sigma\nabla$ is the symmetric part of the covariant derivative in the metric g. Thus we have shown that

$$R_v'(0,g) = 2d. \tag{5.2.13}$$

From (5.2.8) and (5.2.13) we see that

$$F_v'(0,0) = \delta R_v'(0,g) = 2\delta d.$$

As shown in Section 2.4, the Dirichlet problem for the operator δd is elliptic and has zero kernel and co-kernel. Now, the Schauder-type estimates of [2] for elliptic boundary value problems in the spaces $C^{k,\alpha}$ imply that the operator

$$F_v'(0,0) = \delta d : C_0^{k,\alpha}(\tau_M) \to C^{k-2,\alpha}(S^2\tau_M')$$

has a continuous inverse.

We have thus verified that the function (5.2.3) satisfies all conditions of the implicit function theorem [43]. This theorem guarantees local solvability of the equation $F(v,f) = 0$ in v in a neighborhood of the point $(v,f) = (0,0)$, which completes the proof of Theorem 5.2.1.

5.3 Volume and the boundary distance function

The goal of this section is

Theorem 5.3.1 *The volume of a simple Riemannian manifold is uniquely determined by the boundary distance function, i.e., if g^0 and g^1 are two simple metrics on the same compact manifold M, then the equality $\Gamma_{g^0} = \Gamma_{g^1}$ implies that $Vol(M,g^0) = Vol(M,g^1)$.*

First of all we observe that the volume of a CDRM (M,g) can be expressed in terms of the function $\tau_- : \partial_+\Omega M \to \mathbf{R}$ introduced in the definition of a CDRM. Indeed, putting $\varphi \equiv 1$ in the Santalo formula (3.3.4), we obtain

$$\int_M dV^n(x) \int_{\Omega_x M} d\omega_x(\xi) = -\int_{\partial_+\Omega M} \langle \xi, \nu(x)\rangle \tau_-(x,\xi) d\Sigma^{2n-2}(x,\xi).$$

The inner integral on the left-hand side is equal to the volume $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ of the unit sphere in $\mathbf{R}^n$, and we obtain

$$\mathrm{Vol}(M,g) = -\frac{1}{\omega_{n-1}} \int_{\partial_+\Omega M} \langle \xi, \nu(x)\rangle \tau_-(x,\xi) d\Sigma^{2n-2}(x,\xi). \tag{5.3.1}$$

Let now g^0 and g^1 be two simple metrics on the same manifold M such that their boundary distance functions coincide, $\Gamma_{g^0} = \Gamma_{g^1}$. Then these metrics induce the same metric on ∂M, i.e., $\langle \xi, \eta\rangle^0 = \langle \xi, \eta\rangle^1$ for $\xi, \eta \in T(\partial M)$, where $\langle \xi, \eta\rangle^\alpha$ is the inner product with respect to g^α $(\alpha = 0,1)$. Given a point $x \in \partial M$, let $\nu^\alpha(x)$ be the unit vector of the outer normal to the boundary with respect to g^α. Define the linear operator $\mu : T_x M \to T_x M$ such that $\mu|_{T_x(\partial M)} = \mathrm{Id}$ and $\mu\nu^0(x) = \nu^1(x)$. Then μ is an isometry of the Euclidean space $(T_x M, \langle \cdot, \cdot\rangle^0)$ onto $(T_x M, \langle \cdot, \cdot\rangle^1)$, and, consequently, determines the diffeomorphism (denoted by the same letter)

$$\mu : \partial_+\Omega^0 M \to \partial_+\Omega^1 M, \tag{5.3.2}$$

where $\partial_+\Omega^\alpha M = \{(x,\xi) \in TM \mid x \in \partial M, |\xi|^\alpha = 1, \langle \xi, \nu^\alpha(x)\rangle^\alpha \geq 0\}$. By (5.3.1), we can write

$$\mathrm{Vol}\,(M,g^0) = -\frac{1}{\omega_{n-1}} \int_{\partial_+\Omega^0 M} \langle \xi, \nu^0\rangle^0 \tau_-^0 \, d\Sigma^0, \quad \mathrm{Vol}\,(M,g^1) = -\frac{1}{\omega_{n-1}} \int_{\partial_+\Omega^1 M} \langle \xi, \nu^1\rangle^1 \tau_-^1 \, d\Sigma^1,$$

with the function τ^α_- and volume form $d\Sigma^\alpha$ corresponding to the metric g^α. Transforming the second of these integrals with the help of the change of the integration variable which corresponds to the diffeomorphism μ, we obtain

$$\mathrm{Vol}\,(M, g^1) = -\frac{1}{\omega_{n-1}} \int\limits_{\partial_+\Omega^0 M} \langle \xi, \nu^0 \rangle^0 (\tau^1_- \circ \mu)\, \mu^*(d\Sigma^1).$$

Since μ is an isometry, it preserves the volume form, $\mu^*(d\Sigma^1) = d\Sigma^0$. Thus, Theorem 5.3.1 follows from the next statement.

Lemma 5.3.2 *Let g^0 and g^1 be two simple metrics on the same manifold M such that their boundary distance functions coincide, $\Gamma_{g^0} = \Gamma_{g^1}$. Then diffeomorphism (5.3.2) transforms the functions τ^0_- and τ^1_- to each other, i.e., $\tau^1_- \circ \mu = \tau^0_-$.*

The boundary distance function $\Gamma_g(x, y)$ of a simple metric is a smooth function for $x \neq y$. For $x, y \in \partial M$, let $a = \Gamma_g(x, y)$; $\gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) = x$, $\gamma(a) = y$. By the formula for the first variation of the length of a geodesic [33], the next equalities hold:

$$\langle \dot\gamma(0), \xi \rangle = -\partial\Gamma_g(x, y)/\partial\xi \qquad \text{for} \quad \xi \in T_x(\partial M),$$
$$\langle \dot\gamma(a), \xi \rangle = \partial\Gamma_g(x, y)/\partial\xi \qquad \text{for} \quad \xi \in T_y(\partial M),$$

(5.3.3)

which mean that the angles, at which γ intersects ∂M, are uniquely determined by the boundary distance function.

Proof of Lemma 5.3.2. Fix a point $(x, \xi^0) \in \partial_+\Omega^0 M$ and put $\xi^1 = \mu(\xi^0)$. Denote $a^\alpha = \tau^\alpha_-(x, \xi^\alpha)$ $(\alpha = 0, 1)$. We have to prove that $a^0 = a^1$. Let $\gamma^\alpha : [a^\alpha, 0] \to M$ be the geodesic of g^α meeting the initial conditions $\gamma^\alpha(0) = x$ and $\dot\gamma^\alpha(0) = \xi^\alpha$, and $y = \gamma^0(a^0)$. Since g^1 is simple, there exists a geodesic $\widetilde\gamma^1 : [a^0, 0] \to M$ of this metric such that $\widetilde\gamma^1(0) = x$ and $\widetilde\gamma^1(a^0) = y$. The length of the geodesic $\widetilde\gamma^1$ is equal to $\Gamma_{g^1}(x, y) = \Gamma_{g^0}(x, y) = -\tau_-(x, \xi^0) = -a^0$ and, consequently,

$$|\dot{\widetilde\gamma}^1(0)|^1 = |\dot\gamma^1(0)|^1.$$

(5.3.4)

By (5.3.3), the angles between any vector $\eta \in T_x(\partial M)$ and the vectors $\dot\gamma^1(0)$, $\dot{\widetilde\gamma}^1(0)$ are equal. Together with (5.3.4), this gives $\xi^1 = \dot\gamma^1(0) = \dot{\widetilde\gamma}^1(0)$. Both the mappings $\gamma^1 : [a^1, 0] \to M$ and $\widetilde\gamma^1 : [a^0, 0] \to M$ are maximal geodesics of the metric g^1 and satisfy the same initial conditions $\gamma^1(0) = \widetilde\gamma^1(0) = x$ and $\dot\gamma^1(0) = \dot{\widetilde\gamma}^1(0) = \xi^1$. Consequently, they coincide and, in particular, $a^0 = a^1$. The lemma is proved.

5.4 Nonnegativity of the ray transform

The first goal of this section is to show:

Lemma 5.4.1 *If (M, g) is a simple Riemannian manifold then there exists $\varepsilon > 0$ such that if $f \in C^2(S^2\tau'_M)$ satisfies $\|f\|_{C^2(S^2\tau'_M)} < \varepsilon$ and if for every pair $x, y \in \partial M$ the metric $g^1 = g + f$ satisfies $\Gamma_{g^1}(x, y) \geq \Gamma_g(x, y)$, then $If \geq 0$.*

Let (M, g) be a simple Riemannian manifold, and $\varepsilon > 0$ be so small that for every $f \in C^2(S^2\tau'_M)$ such that

$$\|f\|_{C^2(S^2\tau'_M)} < \varepsilon$$

(5.4.1)

the metric

$$g^\tau = g + \tau f \qquad (0 \leq \tau \leq 1)$$

is also simple for every $\tau \in [0, 1]$.

Fix two points $p, q \in \partial M$ and let

$$\gamma_\tau : [0, 1] \to M, \quad \gamma_\tau(0) = p, \quad \gamma_\tau(1) = q$$

be the geodesic of g^τ between p and q. The simplicity of the metrics g^τ guarantees that the γ_τ vary differentiably. Denote the energy of γ_τ by $E(\tau)$:

$$E(\tau) = \int\limits_{\gamma_\tau} g^\tau\, dt = \int\limits_0^1 g^\tau_{ij}(\gamma_\tau(t))\dot\gamma^i_\tau(t)\dot\gamma^j_\tau(t)\, dt.$$

(5.4.2)

Then $E(\tau)$ is a C^2-smooth function on $[0,1]$ and (since $\frac{d}{d\beta}|_{\beta=\tau} \int_{\gamma_\beta} g^\tau \, dt = 0$)

$$E'(\tau) = (I^\tau f)(\gamma_\tau) = \int_0^1 f_{ij}(\gamma_\tau(t))\dot{\gamma}_\tau^i(t)\dot{\gamma}_\tau^j(t) \, dt, \tag{5.4.3}$$

where I^τ is the ray transform in the metric g^τ.

Lemma 5.4.1 now follows from:

Lemma 5.4.2 *The function $E(\tau)$ is concave on $[0,1]$; i.e., $E''(\tau) \leq 0$.*

Proof. Let $0 \leq \tau < \tau' \leq 1$. Since γ_τ is an extremal for g^τ, we can write

$$E(\tau') = \int_{\gamma_{\tau'}} g^{\tau'} \, dt = \int_{\gamma_{\tau'}} (g^\tau + (\tau' - \tau)f) \, dt = \int_{\gamma_{\tau'}} g^\tau \, dt + (\tau' - \tau) \int_{\gamma_{\tau'}} f \, dt$$

$$\geq \int_{\gamma_\tau} g^\tau \, dt + (\tau' - \tau) \int_{\gamma_{\tau'}} f \, dt = E(\tau) + (\tau' - \tau)E'(\tau').$$

Thus,

$$E'(\tau') \leq \frac{E(\tau') - E(\tau)}{\tau' - \tau}. \tag{5.4.4}$$

Similarly

$$E(\tau) = \int_{\gamma_\tau} g^\tau \, dt = \int_{\gamma_\tau} \left(g^{\tau'} - (\tau' - \tau)f\right) \, dt = \int_{\gamma_\tau} g^{\tau'} \, dt - (\tau' - \tau) \int_{\gamma_\tau} f \, dt$$

$$\geq \int_{\gamma_{\tau'}} g^{\tau'} \, dt - (\tau' - \tau) \int_{\gamma_\tau} f \, dt = E(\tau') - (\tau' - \tau)E'(\tau).$$

Thus,

$$E'(\tau) \geq \frac{E(\tau') - E(\tau)}{\tau' - \tau}. \tag{5.4.5}$$

Comparing (5.4.4) and (5.4.5), we obtain $E'(\tau) \geq E'(\tau')$, completing the proof of the lemma.

Taking $\varphi(x,\xi) = f_{ij}(x)\xi^i\xi^j$ in the Santalo formula (3.3.4), we deduce

$$\int_M \left[\int_{\Omega_x M} \xi^i\xi^j \, d\omega_x(\xi)\right] f_{ij}(x) \, dV^n(x) = \int_{\partial_+\Omega M} \langle\xi, \nu(x)\rangle If(x,\xi) \, d\Sigma^{2n-2}(x,\xi). \tag{5.4.6}$$

The left-hand side of this equality is nothing but $(1/n)\lambda$ with $\lambda = (g, f)_{L_2(S^2\tau'_M)}$. We thus arrive at the formula

$$\lambda = n \int_{\partial_+\Omega M} \langle\xi, \nu\rangle If \, d\Sigma^{2n-2}. \tag{5.4.7}$$

Observe that $\lambda = (g, f)_{L_2(S^2\tau'_M)}$ is half of the derivative of the volume of the manifold (M, g^τ) with respect to τ at $\tau = 0$. So we proceed with studying the volume of (M, g^τ).

5.5 Volume of the metric $g^\tau = g + \tau f$

The purpose of this section is to prove:

Lemma 5.5.1 *Let (M, g) be a compact Riemannian manifold with boundary. There exists an $\varepsilon > 0$ such that if $f \in C(S^2\tau'_M)$ satisfies $\|f\|_{C(S^2\tau'_M)} < \varepsilon$ and if $Vol(g + f) \leq Vol(g)$, then*

$$\lambda = (g, f)_{L_2(S^2\tau'_M)} \leq \frac{2}{3}\|f\|^2_{L_2(S^2\tau'_M)}.$$

Proof. We choose a domain $D \subset \mathbf{R}^n$ and a smooth mapping $D \to M$ that carries D diffeomorphically onto an open set of M whose closure coincides with M. Denote the volume of M in the metric $g^\tau = g + \tau f$ by $V(\tau)$. Then

$$V(\tau) = \int_D (\det g^\tau)^{1/2} \, dx. \tag{5.5.1}$$

We represent the integrand of (5.5.1) as follows:

$$\det g^\tau = \det(g + \tau f) = \det g \cdot \det(E + \tau g^{-1} f);$$

$$\det g^\tau = \det g \cdot (1 + \lambda_1 \tau + \lambda_2 \tau^2 + \ldots + \lambda_n \tau^n), \tag{5.5.2}$$

where λ_k is the k-th elementary symmetric function in the eigenvalues $\mu_1, \ldots, \mu_n$ of the matrix $g^{-1} f$. The eigenvalues are real. Note that

$$\langle g, f \rangle = f_i^i = \lambda_1 \tag{5.5.3}$$

and

$$|f|^2 = f_{ij} f^{ij} = \sum_{k=1}^{n} \mu_k^2 = \lambda_1^2 - 2\lambda_2. \tag{5.5.4}$$

Our assumptions and (5.5.4) imply the estimate

$$|\lambda_k| \leq C_k |f|^k \leq C_k \varepsilon^k. \tag{5.5.5}$$

Using the inequality

$$\sqrt{1 + x} \geq 1 + \frac{1}{2}x - \frac{1}{4}x^2 \qquad (|x| \leq \frac{1}{2}),$$

from (5.5.2) we obtain

$$(\det g^\tau)^{1/2} \geq (\det g)^{1/2} \left[1 + \frac{1}{2}(\lambda_1 \tau + \lambda_2 \tau^2 + \ldots + \lambda_n \tau^n) - \frac{1}{4}(\lambda_1 \tau + \lambda_2 \tau^2 + \ldots + \lambda_n \tau^n)^2 \right].$$

With the help of (5.5.5), the last inequality implies the estimate

$$(\det g^\tau)^{1/2} \geq (\det g)^{1/2} \left[1 + \frac{1}{2}\lambda_1 \tau + (\frac{1}{2}\lambda_2 - \frac{1}{4}\lambda_1^2)\tau^2 - C\varepsilon |f|^2 \tau^3 \right]$$

with some constant C depending only on n. Expressing λ_2 through λ_1 and $|f|^2$ by (5.5.4) and inserting the resultant expression in the preceding inequality, we obtain

$$(\det g^\tau)^{1/2} \geq (\det g)^{1/2} \left[1 + \frac{1}{2}\lambda_1 \tau - \frac{1}{4}|f|^2 \tau^2 - C\varepsilon |f|^2 \tau^3 \right].$$

Integrating this inequality over D, we discover that

$$V(\tau) \geq V(0) + \frac{1}{2}\lambda\tau - \frac{1}{4}\|f\|_{L_2}^2 \tau^2 - C\varepsilon\|f\|_{L_2}^2 \tau^3.$$

Since $V(1) \leq V(0)$, this inequality implies that

$$\lambda \leq (\frac{1}{2} + C\varepsilon)\|f\|_{L_2(S^2 \tau'_M)}^2.$$

Choosing an appropriately small ε, we may conclude that

$$\lambda \leq \frac{2}{3}\|f\|_{L_2(S^2 \tau'_M)}^2.$$

5.6   Local estimates for If near $\partial_0\Omega M$

On a CDRM M, definition (3.3.2) of the ray transform and smoothness of the function $\tau_-(x,\xi)$ on $\partial_+\Omega M$ (see Lemma 3.2.1) imply the boundedness of the ray transform in the C^k-norms:

$$\|If\|_{C^k(\partial_+\Omega M)} \le C_k \|f\|_{C^k(S^2\tau_M')}. \tag{5.6.1}$$

The condition that the metrics g and $g+f$ induce the same metric on ∂M is:

$$f_{ij}(x)\xi^i\eta^j = 0 \quad \text{for} \quad x \in \partial M; \quad \xi,\eta \in T_x(\partial M). \tag{5.6.2}$$

Lemma 5.6.1 *If M is a CDRM and a tensor field $f \in C^2(S^2\tau_M')$ satisfies (5.6.2), then the ray transform If vanishes on the boundary $\partial_0\Omega M$ of the manifold $\partial_+\Omega M$ together with all its first-order derivatives.*

Proof. In a neighborhood of a point $x_0 \in \partial M$ we can choose semigeodesic coordinates $(x^1,\ldots,x^n)$ such that x^n coincides with the distance from x to ∂M. In this coordinate system, $g_{in} = \delta_{in}$ and the Christoffel symbols satisfy the relations

$$\Gamma^i_{nn} = \Gamma^n_{in} = 0, \quad \Gamma^\alpha_{\beta n} = -g^{\alpha\gamma}\Gamma^n_{\beta\gamma}.$$

(In this and subsequent formulas, Greek indices vary from 1 to $n-1$; and repeated Greek indices imply the summation from 1 to $n-1$ as usual). The outward unit normal vector ν to ∂M has coordinates $(0,\ldots,0,-1)$, and $\langle \xi,\nu \rangle = -\xi^n = -\xi_n$. The second fundamental form of ∂M

$$\mathrm{II}(\xi,\xi) = \Gamma^n_{\alpha\beta}(x^1,\ldots,x^{n-1},0)\xi^\alpha\xi^\beta$$

is positive definite because of the strict convexity of the boundary. Condition (5.6.2) is written in the chosen coordinates as:

$$f_{\alpha\beta}|_{x^n=0} = 0. \tag{5.6.3}$$

Let $(x^1,\ldots,x^n;\xi^1,\ldots,\xi^n)$ be the associated coordinate system on TM. Then $(x^1,\ldots,x^{n-1};\xi^1,\ldots,\xi^n)$ constitute a local coordinate system on $\partial(TM)$. The submanifold $\partial_+\Omega M$ of $\partial(TM)$ is determined in these coordinates by the relations $g_{ij}(x)\xi^i\xi^j = 1$, $\xi^n = \xi_n \le 0$; and its boundary $\partial_0\Omega M$ is determined by $g_{\alpha\beta}(x)\xi^\alpha\xi^\beta = 1$, $\xi^n = 0$.

The equality

$$\tau_-(x,\xi)|_{\xi_n=0} = 0 \qquad ((x,\xi) \in \partial(TM)) \tag{5.6.4}$$

is evident. We have to prove that

$$If|_{\xi_n=0} = 0, \quad \left.\frac{\partial(If)}{\partial\xi_n}\right|_{\xi_n=0} = 0. \tag{5.6.5}$$

The first of these equalities follows from definition (3.3.2) and (5.6.4). To prove the second one, we rewrite (3.3.2) in the form

$$If(x,\xi) = \int_{\tau_-(x,\xi)}^0 F(t;x,\xi)\,dt, \tag{5.6.6}$$

where

$$F(t;x,\xi) = f_{ij}(\gamma(t;x,\xi))\dot\gamma^i(t;x,\xi)\dot\gamma^j(t;x,\xi). \tag{5.6.7}$$

Differentiating (5.6.6), we obtain

$$\frac{\partial(If)}{\partial\xi_n} = -\frac{\partial\tau_-}{\partial\xi_n}F(\tau_-(x,\xi);x,\xi) + \int_{\tau_-(x,\xi)}^0 \frac{\partial F(t;x,\xi)}{\partial\xi_n}\,dt.$$

Putting $\xi_n = 0$ in this formula and using (5.6.4), we derive

$$\left.\frac{\partial(If)}{\partial\xi_n}\right|_{\xi_n=0} = \left[-\frac{\partial\tau_-}{\partial\xi_n}F\right]_{t=0,\xi_n=0}, \tag{5.6.8}$$

In view of (5.6.3), equality (5.6.7) implies

$$F|_{t=0,\xi_n=0} = f_{\alpha\beta}(x)\xi^\alpha\xi^\beta = 0.$$

This relation together with (5.6.8) implies the second of equalities (5.6.6). The lemma is proved.

Corollary 5.6.2 *Let*

$$L : C^\infty(\partial_+\Omega M) \to C^\infty(\partial_+\Omega M)$$

be a first-order linear differential operator with smooth coefficients on the manifold $\partial_+\Omega M$. If $f \in C^2(S^2\tau'_M)$ is a tensor field satisfying (5.6.2) then the estimate

$$|L(If)(x,\xi)| \leq C\langle \xi, \nu(x)\rangle\|f\|_{C^2(S^2\tau'_M)} \tag{5.6.9}$$

holds with some constant C independent of f.

Proof. For $(x,\xi_1) \in \partial_+\Omega M$, we can choose a curve $t \mapsto \xi_t$, $0 \leq t \leq 1$, in the sphere $\Omega_x M$ which joins ξ_1 with a point ξ_0 such that

$$\langle \xi_0, \nu(x)\rangle = 0, \quad \langle \xi_t, \nu(x)\rangle \geq 0, \quad \left|\frac{d\xi_t}{dt}\right| = \frac{\pi}{2}\langle \xi_1, \nu(x)\rangle.$$

By Lemma 5.6.1, $L(If)(x,\xi_0) = 0$. Therefore,

$$L(If)(x,\xi_1) = \int\limits_0^1 \frac{d}{dt}[L(If)(x,\xi_t)]\, dt.$$

The integral in this formula admits the estimate

$$|L(If)(x,\xi_1)| \leq \int\limits_0^1 |\text{grad}(L(If))(x,\xi_t)|\left|\frac{d\xi_t}{dt}\right|\, dt \leq C\|If\|_{C^2} \cdot \langle \xi_1, \nu(x)\rangle$$

which, together with (5.6.1), gives (5.6.9).

Lemma 5.6.3 *Let M be a CDRM and a tensor field $f \in C^2(S^2\tau'_M)$ satisfy (5.6.2). Fix a semigeodesic coordinate system $(x^1, \ldots, x^n)$ in a neighborhood U of a point $x_0 \in \partial M$ such that $x^n = \text{dist}\,(x, \partial M)$. Then the inequality*

$$|\xi^\alpha \overset{v}{\nabla}_n \overset{h}{\nabla}_\alpha(If)(x,\xi)| \leq C\|f\|_{C^2} \cdot \langle \xi, \nu(x)\rangle \tag{5.6.10}$$

holds for all $x \in U \cap \partial M$ with a constant C independent of f. Here the summation from 1 to $n-1$ is meant with respect to the index α.

Proof. The left-hand side of (5.6.10) can be written as follows:

$$\xi^\alpha \overset{v}{\nabla}_n \overset{h}{\nabla}_\alpha(If) = \xi^\alpha \frac{\partial^2(If)}{\partial x^\alpha \partial \xi^n} + L(If),$$

where L is a first order linear differential operator on $\partial_+\Omega M$. On taking Corollary 5.6.2 into account, estimate (5.6.10) follows from the inequality

$$\left|\frac{\partial^2 If(x,\xi)}{\partial x^\alpha \partial \xi^n}\right| \leq C\langle \xi, \nu(x)\rangle\|f\|_{C^2}. \tag{5.6.11}$$

So our goal is proving estimate (5.6.11).

Differentiating (5.6.6), we obtain

$$\frac{\partial^2(If(x,\xi))}{\partial x^\alpha \partial \xi^n} = -\frac{\partial^2\tau_-(x,\xi)}{\partial x^\alpha \partial \xi^n}F(\tau_-(x,\xi); x,\xi) - \frac{\partial\tau_-(x,\xi)}{\partial x^\alpha}\frac{\partial}{\partial \xi^n}[F(\tau_-(x,\xi); x,\xi)]$$

$$-\frac{\partial\tau_-(x,\xi)}{\partial \xi^n}\frac{\partial F}{\partial x^\alpha}(\tau_-(x,\xi); x,\xi) + \int\limits_{\tau_-(x,\xi)}^0 \frac{\partial^2 F(t; x,\xi)}{\partial x^\alpha \partial \xi^n}\, dt. \tag{5.6.12}$$

Introducing the notation

$$\varphi(x,\xi) = F(\tau_-(x,\xi); x,\xi), \tag{5.6.13}$$

we have

$$\frac{\partial F}{\partial x^\alpha}(\tau_-(x,\xi); x,\xi) = \frac{\partial\varphi(x,\xi)}{\partial x^\alpha} - \frac{\partial F}{\partial t}(\tau_-(x,\xi); x,\xi)\frac{\partial\tau_-(x,\xi)}{\partial x^\alpha}.$$

Substituting this expression into (5.6.12), we obtain

$$\frac{\partial^2 (If(x,\xi))}{\partial x^\alpha \partial \xi^n} = -\frac{\partial^2 \tau_-(x,\xi)}{\partial x^\alpha \partial \xi^n} \varphi(x,\xi) - \frac{\partial \tau_-(x,\xi)}{\partial x^\alpha} \left(\frac{\partial \varphi(x,\xi)}{\partial \xi^n} + \frac{\partial F}{\partial t}(\tau_-(x,\xi); x,\xi) \right)$$

$$- \frac{\partial \tau_-(x,\xi)}{\partial \xi^n} \frac{\partial \varphi(x,\xi)}{\partial x^\alpha} + \int\limits_{\tau_-(x,\xi)}^{0} \frac{\partial^2 F(t; x,\xi)}{\partial x^\alpha \partial \xi^n} \, dt. \qquad (5.6.14)$$

Formulas (5.6.7) and (5.6.13) imply the estimates

$$\|F\|_{C^k} \leq C\|f\|_{C^k}, \quad \|\varphi\|_{C^k} \leq C\|f\|_{C^k}$$

for every k. Therefore (5.6.14) implies the inequality

$$\left| \frac{\partial^2 If(x,\xi)}{\partial x^\alpha \partial \xi^n} \right| \leq C \left(|\varphi(x,\xi)| + \left| \frac{\partial \varphi(x,\xi)}{\partial x^\alpha} \right| + \left| \frac{\partial \tau_-(x,\xi)}{\partial x^\alpha} \right| \|f\|_{C^1} + |\tau_-(x,\xi)| \cdot \|f\|_{C^2} \right).$$

The latter inequality would imply estimate (5.6.11) if we demonstrate that

$$|\tau_-(x,\xi)| \leq C\langle \xi, \nu(x) \rangle, \quad \frac{\partial \tau_-(x,\xi)}{\partial x^\alpha} \leq C\langle \xi, \nu(x) \rangle, \qquad (5.6.15)$$

$$|\varphi(x,\xi)| \leq C\langle \xi, \nu(x) \rangle \|f\|_{C^1}, \quad \left| \frac{\partial \varphi(x,\xi)}{\partial x^\alpha} \right| \leq C\langle \xi, \nu(x) \rangle \|f\|_{C^2}. \qquad (5.6.16)$$

Estimates (5.6.15) are evident because $\tau_-(x,\xi)$ and $\partial \tau_-(x,\xi)/\partial x^\alpha$ are smooth functions on $\partial_+ \Omega M$ vanishing on the boundary $\partial_0 \Omega M$ which is determined by the equation $\langle \xi, \nu(x) \rangle = 0$.

To prove estimates (5.6.16) we first note that the function $\varphi(x,\xi)$ (and, consequently, $\partial \varphi(x,\xi)/\partial x^\alpha$) vanishes on $\partial_0 \Omega M$. Indeed, $\tau_-(x,\xi) = 0$ for $(x,\xi) \in \partial_0 \Omega M$, and definitions (5.6.7) and (5.6.13) give us

$$\varphi(x,\xi) = f_{ij}(x)\xi^i \xi^j.$$

Since $f_{\alpha\beta}(x) = 0$ $(1 \leq \alpha, \beta \leq n-1)$ and $\xi^n = 0$, this implies that $\varphi(x,\xi) = 0$.

Given a point $(x,\xi_1) \in \partial_+ \Omega M$, we can join it with a point $(x,\xi_0) \in \partial_0 \Omega M$ by a curve $(x,\xi_t) \in \partial_+ \Omega M$ $(0 \leq t \leq 1)$ such that $|d\xi_t/dt| = \langle \xi_1, \nu(x) \rangle$. Using the representations

$$\varphi(x,\xi_1) = \int\limits_0^1 \frac{d}{dt} \left(\varphi(x,\xi_t) \right) dt, \quad \frac{\partial \varphi(x,\xi_1)}{\partial x^\alpha} = \int\limits_0^1 \frac{d}{dt} \left(\frac{\partial \varphi}{\partial x^\alpha}(x,\xi_t) \right) dt,$$

we obtain the estimates

$$|\varphi(x,\xi_1)| \leq \int\limits_0^1 \|\varphi\|_{C^1} \left| \frac{d\xi_t}{dt} \right| dt \leq C\langle \xi_1, \nu(x) \rangle \|f\|_{C^1},$$

$$\left| \frac{\partial \varphi(x,\xi_1)}{\partial x^\alpha} \right| \leq \int\limits_0^1 \|\varphi\|_{C^2} \left| \frac{d\xi_t}{dt} \right| dt \leq C\langle \xi_1, \nu(x) \rangle \|f\|_{C^2}$$

that are equivalent to (5.6.16). The lemma is proved.

5.7   Proof of Theorem 5.1.1

To prove Theorem 5.1.1, it is sufficient to prove Lemma 5.1.2. Thus we let (M, g) satisfy the hypotheses of Lemma 5.1.2 and let g^1 be a metric $C^{3,\alpha}$-close enough to g and such that the boundary distance-functions satisfy $d_{g^1}(x,y) \geq d_g(x,y)$ for all $x, y \in \partial M$, the induced Riemannian metrics on ∂M coincide, and $\mathrm{Vol}(g^1) \leq \mathrm{Vol}(g)$. We will show that g^1 is isometric to g. In view of Theorem 5.2.1, we may assume that the tensor field $f = g^1 - g$ is solenoidal and satisfies the inequality $\|f\|_{C^2(S^2 \tau'_M)} < \varepsilon$ with an arbitrary small $\varepsilon > 0$. By choosing ε sufficiently small and applying Lemma 5.4.1, equation (5.4.7), and Lemma 5.5.1 we see that the tensor field f satisfies:

$$\delta f = 0, \tag{5.7.1}$$

$$\|f\|_{C^2(S^2\tau'_M)} < \varepsilon, \tag{5.7.2}$$

$$If \geq 0, \tag{5.7.3}$$

$$n \int_{\partial_+\Omega M} \langle \xi, \nu \rangle If \, d\Sigma^{2n-2} = \lambda \leq \frac{2}{3}\|f\|^2_{L_2(S^2\tau'_M)}. \tag{5.7.4}$$

We will prove that $f = 0$.

Given f, we define the function $u \in C^2(T^0M \setminus T(\partial M))$ by the equality

$$u(x,\xi) = \int_{\tau_-(x,\xi)}^{0} f_{ij}(\gamma_{x,\xi}(t))\dot{\gamma}^i_{x,\xi}(t)\dot{\gamma}^j_{x,\xi}(t)\, dt \qquad ((x,\xi) \in T^0M). \tag{5.7.5}$$

This function satisfies the boundary conditions

$$u|_{\partial_-\Omega M} = 0$$

and

$$u(x,\xi) = If(x,\xi) \geq 0 \quad \text{for} \quad (x,\xi) \in \partial_+\Omega M. \tag{5.7.6}$$

The inequality (5.7.6) is just (5.7.3).

Since f is solenoidal, the Pestov integral identity (4.3.10) for the function u is:

$$\int_{\Omega M} \left[|\overset{h}{\nabla}u|^2 - R_{ijkl}\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u + (n+2)|Hu|^2\right] d\Sigma = \int_{\partial_+\Omega M} \left[L(If) - 4(If)f_{ij}\xi^i\nu^j\right] d\Sigma^{2n-2},$$

where L is the quadratic first-order differential operator on the manifold $\partial_+\Omega M$ which is expressed in semigeodesic coordinates $(x^1, \ldots, x^{n-1}, x^n = $ distance to the boundary$)$ by formula (4.3.9).

We introduce the semibasic covector fields y and z by formulas (4.3.12) and (4.3.13), and rewrite the latter equality in the form

$$\int_{\Omega M} |z|^2 d\Sigma + (n+3) \int_{\Omega M} (Hu)^2 d\Sigma = \int_{\Omega M} R_{ijkl}\xi^i\xi^k y^j y^l d\Sigma + \int_{\partial_+\Omega M} \left[L(If) - 4(If)f_{ij}\xi^i\nu^j\right] d\Sigma^{2n-2}.$$

Estimating the first integral on the right-hand side with the help of (4.3.23), we arrive at the inequality

$$(1-k^+)\int_{\Omega M} |z|^2 d\Sigma + [(n+3) - 2(n+2)k^+]\int_{\Omega M} (Hu)^2 d\Sigma \leq \int_{\partial_+\Omega M} \left[L(If) - 4(1+k^+)(If)f_{ij}\xi^i\nu^j\right] d\Sigma^{2n-2}. \tag{5.7.7}$$

The heart of the rest of the proof is:

Lemma 5.7.1 *There is a constant C independent of f and such that*

$$\int_{\partial_+\Omega M} \left[L(If) - 4(1+k^+)(If)f_{ij}\xi^i\nu^j\right] d\Sigma^{2n-2} \leq C\|f\|_{C^2(S^2\tau'_M)}\lambda.$$

We will come back to the proof of this lemma but we first show how the theorem will follow.
Lemma 5.7.1, along with (5.7.7), gives

$$(1-k^+)\int_{\Omega M} |z|^2 d\Sigma + [(n+3) - 2(n+2)k^+]\int_{\Omega M} (Hu)^2 d\Sigma \leq C\|f\|_{C^2}\lambda. \tag{5.7.8}$$

Using (5.7.2) and (5.7.4), we obtain

$$(1-k^+)\int_{\Omega M} |z|^2 d\Sigma + [(n+3) - 2(n+2)k^+]\int_{\Omega M} (Hu)^2 d\Sigma \leq C\varepsilon\|f\|^2_{L_2}. \tag{5.7.9}$$

Finally, the kinetic equation $Hu = f_{ij}(x)\xi^i\xi^j$ implies the estimate

$$\|f\|_{L_2}^2 \leq C' \int\limits_{\Omega M} (Hu)^2 \, d\Sigma \tag{5.7.10}$$

with some constant C' independent of f. Combining (5.7.9) and (5.7.10), we arrive at the final estimate

$$(1 - k^+) \int\limits_{\Omega M} |z|^2 \, d\Sigma + [(n+3) - 2(n+2)k^+ - CC'\varepsilon] \int\limits_{\Omega M} (Hu)^2 \, d\Sigma \leq 0. \tag{5.7.11}$$

Since we can choose $\varepsilon > 0$ arbitrarily small, we can choose it so that the coefficients of both integrals in (5.7.11) are positive. Therefore (5.7.11) implies that $Hu \equiv 0$ and hence $f \equiv 0$. The theorem is proved.

Proof of Lemma 5.7.1. First we transform the integral $\int\limits_{\partial_+\Omega M} Lu \, d\Sigma^{2n-2}$ by integration by parts. To this end we rewrite (4.3.9) as follows:

$$Lu = a^i \overset{v}{\nabla}_i u, \tag{5.7.12}$$

where

$$a^\alpha = \xi^n \overset{h}{\nabla}^\alpha u, \quad a^n = -\xi^\alpha \overset{h}{\nabla}_\alpha u. \tag{5.7.13}$$

We extract a divergent term from (5.7.12):

$$Lu = \overset{v}{\nabla}_i(ua^i) - u\overset{v}{\nabla}_i a^i = \overset{v}{\nabla}_i(ua^i) - \xi^n u \overset{v}{\nabla}_\alpha \overset{h}{\nabla}^\alpha u + u\overset{v}{\nabla}_n(\xi^\alpha \overset{h}{\nabla}_\alpha u).$$

Integrating this equality over $\partial_+\Omega M$ and transforming the first term by Gauss — Ostrogradskiĭ, we obtain

$$\int\limits_{\partial_+\Omega M} Lu \, d\Sigma^{2n-2} = k \int\limits_{\partial_+\Omega M} u\langle \xi, a \rangle \, d\Sigma^{2n-2} - \int\limits_{\partial_+\Omega M} \left[\langle \xi, \nu \rangle u\overset{v}{\nabla}_\alpha \overset{h}{\nabla}^\alpha u - u\overset{v}{\nabla}_n(\xi^\alpha \overset{h}{\nabla}_\alpha u) \right] d\Sigma^{2n-2}.$$

The coefficient k depends on the degree of homogeneity of a. Its value does not matter because $\langle \xi, a \rangle = 0$ as we see from (5.7.13). Consequently,

$$\int\limits_{\partial_+\Omega M} Lu \, d\Sigma^{2n-2} = - \int\limits_{\partial_+\Omega M} \left[\langle \xi, \nu \rangle u\overset{v}{\nabla}_\alpha \overset{h}{\nabla}^\alpha u - u\xi^\alpha \overset{v}{\nabla}_n \overset{h}{\nabla}_\alpha u \right] d\Sigma^{2n-2}.$$

We thus see that

$$\int\limits_{\partial_+\Omega M} \left[L(If) - 4(1 + k^+)(If)f_{ij}\xi^i\nu^j \right] d\Sigma^{2n-2} =$$

$$= - \int\limits_{\partial_+\Omega M} \langle \xi, \nu \rangle If \cdot \overset{v}{\nabla}_\alpha \overset{h}{\nabla}^\alpha (If) \, d\Sigma^{2n-2} + \int\limits_{\partial_+\Omega M} If \cdot \xi^\alpha \overset{v}{\nabla}_n \overset{h}{\nabla}_\alpha (If) \, d\Sigma^{2n-2} - 4(1+k^+) \int\limits_{\partial_+\Omega M} If \cdot f_{ij}\xi^i\nu^j \, d\Sigma^{2n-2}.$$

$$\tag{5.7.14}$$

Some terms in this equation are written by using local coordinates. Nevertheless, all the integrands are invariant; i.e., they are independent of the choice of coordinates.

We will now estimate each of the integrals on the right-hand side of (5.7.14).

Using the nonnegativity of If, we obtain

$$\left| \int\limits_{\partial_+\Omega M} \langle \xi, \nu \rangle If \cdot \overset{v}{\nabla}_\alpha \overset{h}{\nabla}^\alpha (If) \, d\Sigma^{2n-2} \right| \leq \|If\|_{C^2} \int\limits_{\partial_+\Omega M} \langle \xi, \nu \rangle If \, d\Sigma^{2n-2}.$$

Together with (5.7.4) and (5.6.1), this gives

$$\left| \int\limits_{\partial_+\Omega M} \langle \xi, \nu \rangle If \cdot \overset{v}{\nabla}_\alpha \overset{h}{\nabla}^\alpha (If) \, d\Sigma^{2n-2} \right| \leq C\|f\|_{C^2}\lambda \tag{5.7.15}$$

with some constant C independent of f.

Applying Lemma 5.6.3, we obtain

$$\left| \int_{\partial_+\Omega M} If \cdot \xi^\alpha \overset{v}{\nabla}_n \overset{h}{\nabla}_\alpha(If) \, d\Sigma^{2n-2} \right| \leq C\|f\|_{C^2} \int_{\partial_+\Omega M} \langle \xi, \nu \rangle If \, d\Sigma^{2n-2} = C\|f\|_{C^2}\lambda. \tag{5.7.16}$$

The estimation of the last integral on the right-hand side of (5.7.14),

$$J = \int_{\partial_+\Omega M} If \cdot f_{ij}\xi^i\nu^j \, d\Sigma^{2n-2}, \tag{5.7.17}$$

is more troublesome because the factor $f_{ij}\xi^i\nu^j$ of its integrand does not vanish on $\partial_0\Omega M$ (more precisely, we are not able to prove that it vanishes a priori). To estimate this integral, we introduce the mapping

$$\Phi : \partial_+\Omega M \to \partial_+\Omega M \tag{5.7.18}$$

by putting

$$\Phi(x,\xi) = (y,\eta), \quad \text{where} \quad y = \gamma_{x,\xi}(\tau_-(x,\xi)), \quad \eta = -\dot{\gamma}_{x,\xi}(\tau_-(x,\xi)). \tag{5.7.19}$$

It is evident that Φ is smooth and $\Phi^2 = \text{Id}$. Consequently, Φ is a diffeomorphism. One can see (Lemma of [17]) by a double use of Santaló's formula and the fact that the map $v \mapsto -v$ is measure preserving on ΩM that the absolute value of the Jacobian on $\partial_+\Omega M \setminus \partial_0\Omega M$ of Φ is $\frac{\langle \xi, \nu(x) \rangle}{\langle \eta, \nu(y) \rangle}$; i.e.,

$$\frac{\langle \xi, \nu(x) \rangle}{\langle \eta, \nu(y) \rangle} = \left| \frac{d\Sigma^{2n-2}(y,\eta)}{d\Sigma^{2n-2}(x,\xi)} \right|. \tag{5.7.20}$$

We need to study what happens on the boundary $\partial_0\Omega M$ of $\partial_+\Omega M$. The relations

$$y(x,\xi) = x, \quad \eta(x,\xi) = -\xi \quad \text{for} \quad (x,\xi) \in \partial_0\Omega M \tag{5.7.21}$$

are evident. We will use the same semigeodesic coordinates as in the proof of Lemma 5.6.1. We will now find the derivative $\left. \frac{\partial \tau_-(x,\xi)}{\partial \xi_n} \right|_{\xi_n=0}$. To this end, given a point $(x,\xi) \in \partial(TM), \xi \neq 0$, we denote by

$$\gamma(t;x,\xi) = (\gamma^1(t;x,\xi),\ldots,\gamma^n(t;x,\xi))$$

the geodesic in M that satisfies the initial conditions

$$\gamma^\alpha(0;x,\xi) = x^\alpha, \quad \gamma^n(0;x,\xi) = 0, \quad \dot{\gamma}^i(0;x,\xi) = \xi^i.$$

The equation

$$\gamma^n(t;x,\xi) = 0$$

has two solutions $t = 0$ and $t = \tau_-(x,\xi)$. Representing the function $\gamma^n(t;x,\xi)$ in the form

$$\gamma^n(t;x,\xi) = \xi_n t - \frac{1}{2}\Gamma^n_{\alpha\beta}(x)\xi^\alpha\xi^\beta t^2 + \varphi(t;x,\xi)t^3$$

with some smooth function $\varphi(t;x,\xi)$, we see that $\tau_-(x,\xi)$ satisfies the equation

$$\xi_n - \frac{1}{2}\Gamma^n_{\alpha\beta}(x)\xi^\alpha\xi^\beta\tau_-(x,\xi) + \varphi(\tau_-(x,\xi);x,\xi)\tau_-^2(x,\xi) = 0.$$

Differentiating this equation with respect to ξ_n and putting $\xi_n = 0$, we obtain

$$\left. \frac{\partial \tau_-(x,\xi)}{\partial \xi_n} \right|_{\xi_n=0} = 2\left(\Gamma^n_{\alpha\beta}(x)\xi^\alpha\xi^\beta\right)^{-1}. \tag{5.7.22}$$

Differentiating (5.7.19) and using (5.6.4) and (5.7.22), we obtain on $\partial_0\Omega M$

$$\frac{\partial \eta^n}{\partial \xi_n} = -1 - \ddot{\gamma}^n \frac{\partial \tau_-}{\partial \xi_n} = -1 + 2\Gamma^n_{\alpha\beta}\xi^\alpha\xi^\beta(\Gamma^n_{\alpha\beta}\xi^\alpha\xi^\beta)^{-1} = 1.$$

Since $\langle\xi,\nu\rangle = -\xi^n$ and $\langle\eta,\nu\rangle = -\eta^n$ in the chosen coordinates, the latter relation implies the representation

$$\langle\eta,\nu\rangle = \frac{\langle\xi,\nu\rangle}{1 + \langle\xi,\nu\rangle\varphi} \tag{5.7.23}$$

with some $\varphi \in C^\infty(\partial_+\Omega M)$. In particular, this means that the absolute value of the Jacobian of Φ (i.e. $\frac{\langle\xi,\nu\rangle}{\langle\eta,\nu\rangle}$) goes to 1 as (x,ξ) approaches $\partial_0\Omega M$.

We observe that the ray transform is invariant under Φ, i.e.,

$$If(y,\eta) = If(x,\xi) \quad \text{for} \quad (y,\eta) = \Phi(x,\xi).$$

Using the change of variables $(y,\eta) = \Phi(x,\xi)$ in (5.7.17), we obtain

$$J = \int\limits_{\partial_+\Omega M} (If)(y,\eta) \cdot f_{ij}(y)\eta^i\nu^j(y)\, d\Sigma^{2n-2}(y,\eta) =$$

$$= \int\limits_{\partial_+\Omega M} (If)(x,\xi) \cdot f_{ij}(y(x,\xi))\eta^i(x,\xi)\nu^j(y(x,\xi))\frac{\langle\xi,\nu\rangle}{\langle\eta,\nu\rangle}\, d\Sigma^{2n-2}(x,\xi).$$

Adding this equality to (5.7.17), we obtain

$$2J = \int\limits_{\partial_+\Omega M} (If)(x,\xi)\langle\xi,\nu\rangle\left[\frac{f_{ij}(y(x,\xi))\eta^i(x,\xi)\nu^j(y(x,\xi))}{\langle\eta,\nu\rangle} + \frac{f_{ij}(x)\xi^i\nu^j(x)}{\langle\xi,\nu\rangle}\right] d\Sigma^{2n-2}(x,\xi). \tag{5.7.24}$$

We now need to bound the expression in brackets from above. We rewrite it as the sum of two terms:

$$2J = \int\limits_{\partial_+\Omega M} (If)(x,\xi)\langle\xi,\nu\rangle(A + B)d\Sigma^{2n-2}(x,\xi), \tag{5.7.25}$$

where:

$$A = \frac{f_{ij}(y(x,\xi))\eta^i(x,\xi)\nu^j(y(x,\xi)) + f_{ij}(x)\xi^i\nu^j(x)}{\langle\eta,\nu\rangle}$$

and

$$B = f_{ij}(x)\xi^i\nu^j(x)\left[\frac{1}{\langle\xi,\nu\rangle} - \frac{1}{\langle\eta,\nu\rangle}\right].$$

Equations (5.7.23) and the compactness of M tell us that the part in brackets of B is uniformly bounded by a constant independent of f. So B is bounded by a constant times $\|f\|_{C^0}$.

Away from $\partial_0\Omega M$ (i.e. $\langle\xi,\nu\rangle \geq \text{constant} > 0$) A is clearly bounded by a constant times $\|f\|_{C^0}$. Near the boundary the numerator of A is bounded by $\|\nabla f\|\rho(x,y(x,\xi)) + \|f\|_{C^0}|\eta + \xi|$ (where we interpret $\eta + \xi$ as the vector at x with coordinates $(\xi^i + \eta^i)$) . Now by Lemma 3.2.2, $\rho(x,y(x,\xi)) = |\tau_-(x,\xi)| \leq$ constant $\langle\xi,\nu(x)\rangle$ and hence the relations (5.7.23) tell us that near the boundary A is bounded by a constant times $\|f\|_{C^1}$. Thus we see that A is bounded on all of $\partial_+\Omega M$ by a constant times $\|f\|_{C^1}$.

Combining these estimates with equation (5.7.25) we get

$$|J| \leq C\|f\|_{C^1} \int\limits_{\partial_+\Omega M} \langle\xi,\nu\rangle If\, d\Sigma^{2n-2} = C\|f\|_{C^1}\lambda. \tag{5.7.26}$$

Lemma 5.7.1 (and hence Theorem 5.1.1) now follows by combining (5.7.15), (5.7.16) and (5.7.26).

Lecture 6
The modified horizontal derivative

Here we present some new version of the Pestov identity that does not contain explicitly the curvature tensor. This identity allows us to strengthen Theorem 3.4.3, in the cases of $m = 0$ and $m = 1$, by replacing assumption (3.4.4) on curvature with the assumption of simplicity of the metric. In the case of $m \geq 2$, we obtain also a new finiteness result on the ray transform on a simple Riemannian manifold.

6.1 The modified horizontal derivative

Before formal presentation, we will informally discuss the main idea that leads to the notion of the modified derivative.

Let (M, g) be a Riemannian manifold. Consider the simplest kinetic equation

$$Hu(x, \xi) = f(x) \qquad (6.1.1)$$

on the manifold TM.

In the proof of Theorem 3.4.3, a key role was played by the Pestov identity:

$$2 \langle \overset{h}{\nabla} u, \overset{v}{\nabla}(Hu) \rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_i v^i + \overset{v}{\nabla}_i w^i - R_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u, \qquad (6.1.2)$$

where (u^i) and (v^i) are some semibasic vector fields expressible in terms of the function $u \in C^\infty(TM)$, whose specific form is however irrelevant here. We raise the question: are there any modifications of the Pestov identity (6.1.2) appropriate for studying inverse problems for equation (6.1.1) as well as identity (6.1.2) itself?

The application of the operator $\overset{v}{\nabla}$ to equation (6.1.1) seems justified, since the operator annihilates the right-hand side of the equation. On the other hand, the scalar multiplication of the vector $\overset{v}{\nabla}(Hu)$ just by $\overset{h}{\nabla} u$ on the left-hand side of identity (6.1.2) is not dictated by equation (6.1.1). A motivation of such multiplication is mostly due to the form of the right-hand side of (6.1.2) that consists of the positive definite quadratic form $|\overset{h}{\nabla} u|^2$ and the terms $\overset{h}{\nabla}_i v^i$ and $\overset{v}{\nabla}_i w^i$ of divergence type. The right-hand side of (6.1.2) also contains the fourth summand $R_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u$, but it is the very term that is undesirable for us. Using a rather remote analogy, our observation can be expressed by the following figurative, although nonrigorous statement: $\overset{h}{\nabla} u$ is an integrating factor for the equation $\overset{v}{\nabla}(Hu) = 0$. Does this equation admit other integrating factors?

To answer the last question, let us analyze the proof of the Pestov identity in Section 4.1. Which properties of the horizontal derivative were used in the proof? First of all, these are commutation formulas for the operators $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$. As regards the properties of the horizontal derivative listed in Theorem 2.6.1, they were in no way used to a full extend. Namely, the first of the mentioned properties (the agreement of $\overset{h}{\nabla}$ and ∇ on basic tensor fields) was not involved at all, and instead of the second one the following weaker claim was used:

$$Hu = \xi^i \overset{h}{\nabla}_i u. \qquad (6.1.3)$$

If we drop the first property and replace the second one with (6.1.3) in the above theorem, then the assertion of the theorem concerning uniqueness of the operator $\overset{h}{\nabla}$ becomes false. It turns out that in this case the operator $\overset{h}{\nabla}$ is determined to within an arbitrary semibasic tensor field of degree 2. This liberty

can be used to compensate the last term on the right-hand side of (6.1.2), which constitutes the main idea of the current lecture.

Now we turn to formal presentation. Let (M, g) be a Riemannian manifold. Fix a semibasic tensor field $a = (a^{ij}) \in C^\infty(\beta_0^2 M)$ which is symmetric:

$$a^{ij} = a^{ji}; \tag{6.1.4}$$

positively homogeneous in its second argument:

$$a^{ij}(x, \lambda\xi) = \lambda a^{ij}(x, \xi) \quad (\lambda > 0); \tag{6.1.5}$$

and orthogonal to the vector ξ:

$$a^{ij}(x, \xi)\xi_j = 0. \tag{6.1.6}$$

Using the field a, define the *modified horizontal derivative*

$$\overset{a}{\nabla} : C^\infty(\beta_s^r M) \to C^\infty(\beta_s^{r+1} M) \tag{6.1.7}$$

with the *modifying tensor a* by the equality

$$\overset{a}{\nabla} u = \overset{a}{\nabla}{}^k u^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial \xi^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial \xi^{i_r}} \otimes \frac{\partial}{\partial \xi^k} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}, \tag{6.1.8}$$

where

$$\overset{a}{\nabla}{}^k u^{i_1 \dots i_r}_{j_1 \dots j_s} = \overset{h}{\nabla}{}^k u^{i_1 \dots i_r}_{j_1 \dots j_s} + A^k u^{i_1 \dots i_r}_{j_1 \dots j_s} \tag{6.1.9}$$

and

$$A^k u^{i_1 \dots i_r}_{j_1 \dots j_s} = a^{kp} \overset{v}{\nabla}_p u^{i_1 \dots i_r}_{j_1 \dots j_s} - \sum_{\alpha=1}^r \overset{v}{\nabla}_p a^{k i_\alpha} \cdot u^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} + \sum_{\alpha=1}^s \overset{v}{\nabla}_{j_\alpha} a^{kp} \cdot u^{i_1 \dots i_r}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_s}. \tag{6.1.10}$$

The so-defined operator $\overset{a}{\nabla}$ is obviously independent of the choice of an associated coordinate system involved in formulas (6.1.8)–(6.1.10). Consider its main properties.

First of all, $\overset{a}{\nabla}$ commutes with the contraction operators C_l^k and is a derivative with respect to the tensor product in the sense that

$$\overset{a}{\nabla}(u \otimes v) = \rho^{r+1}(\overset{a}{\nabla} u \otimes v) + u \otimes \overset{a}{\nabla} v \quad (u \in C^\infty(\beta_s^r(M)),$$

where ρ^{r+1} is the permutation of upper indices which translates the $(r+1)$th index to the final position (cf. Theorem 2.6.1). Both the properties are verified by direct calculation in coordinates and thus omitted.

Let us clarify the interrelation between $\overset{a}{\nabla}$ and H. To this end, we multiply equality (6.1.10) by ξ_k and sum over k. By making use of (6.1.5) and (6.1.6), we obtain

$$\xi_k A^k u^{i_1 \dots i_r}_{j_1 \dots j_s} = -\sum_{\alpha=1}^r \xi_k \overset{v}{\nabla}_p a^{k i_\alpha} \cdot u^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} + \sum_{\alpha=1}^s \xi_k \overset{v}{\nabla}_{j_\alpha} a^{kp} \cdot u^{i_1 \dots i_r}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_s}$$

$$= \sum_{\alpha=1}^r a^{i_\alpha}_p u^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} - \sum_{\alpha=1}^s a^p_{j_m} u^{i_1 \dots i_r}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_s}.$$

Here the usual rule $a^i_j = g_{jk} a^{ik}$ of lowering indices is used. Consequently,

$$\xi_k \overset{a}{\nabla}{}^k u^{i_1 \dots i_r}_{j_1 \dots j_s} = (Hu)^{i_1 \dots i_r}_{j_1 \dots j_s} + \sum_{\alpha=1}^r a^{i_m}_p u^{i_1 \dots i_{\alpha-1} p i_{\alpha+1} \dots i_r}_{j_1 \dots j_s} - \sum_{\alpha=1}^s a^p_{j_\alpha} u^{i_1 \dots i_r}_{j_1 \dots j_{\alpha-1} p j_{\alpha+1} \dots j_s}. \tag{6.1.11}$$

In particular, for a scalar function $u \in C^\infty(TM)$, we have

$$Hu = \xi_i \overset{a}{\nabla}{}^i u. \tag{6.1.12}$$

From (6.1.6) and the equality $\xi^p \overset{v}{\nabla}_p a^{ij} = a^{ij}$ following from (6.1.5), we obtain

$$\overset{a}{\nabla}{}^i \xi^j = \overset{a}{\nabla}{}^i \xi_j = 0. \tag{6.1.13}$$

In the case of a general field a, the metric tensor is not parallel with respect to $\overset{a}{\nabla}$. Indeed,

$$\overset{a}{\nabla}{}^i g_{jk} = \overset{v}{\nabla}_j a^{ip} \cdot g_{pk} + \overset{v}{\nabla}_k a^{ip} \cdot g_{jp} = \overset{v}{\nabla}_j a^i_k + \overset{v}{\nabla}_k a^i_j.$$

In view of this fact, care should be exercised while raising and lowering indices in expressions containing $\overset{a}{\nabla}$. Just for this reason, we originally preferred to define the operator $\overset{a}{\nabla}{}^i$ with a superscript.

We will obtain a commutation formula for the operators $\overset{a}{\nabla}$ and $\overset{v}{\nabla}$. Using commutability of $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$, we have

$$(\overset{a}{\nabla}{}^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j\overset{a}{\nabla}{}^i)u^{i_1\ldots i_r}_{j_1\ldots j_s} = (A^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j A^i)u^{i_1\ldots i_r}_{j_1\ldots j_s}.$$

Transforming the right-hand side of this equality in accord with definition (6.1.10), after simple calculations we arrive at the formula

$$(\overset{a}{\nabla}{}^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j\overset{a}{\nabla}{}^i)u^{i_1\ldots i_r}_{j_1\ldots j_s} = \sum_{\alpha=1}^{r} \overset{v}{\nabla}_j\overset{v}{\nabla}_p a^{ii_\alpha} \cdot u^{i_1\ldots i_{\alpha-1}pi_{\alpha+1}\ldots i_r}_{j_1\ldots j_s} - \sum_{\alpha=1}^{s} \overset{v}{\nabla}_j\overset{v}{\nabla}_{j_\alpha} a^{ip} \cdot u^{i_1\ldots i_r}_{j_1\ldots j_{\alpha-1}pj_{\alpha+1}\ldots j_s}. \quad (6.1.14)$$

In particular, for a scalar function $u \in C^\infty(TM)$, we have

$$(\overset{a}{\nabla}{}^i\overset{v}{\nabla}_j - \overset{v}{\nabla}_j\overset{a}{\nabla}{}^i)u = 0. \qquad (6.1.15)$$

Let us obtain a commutation formula for $\overset{a}{\nabla}{}^i$ and $\overset{a}{\nabla}{}^j$. First, we consider the case of a scalar function $u \in C^\infty(TM)$. By (6.1.9), we have

$$(\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j - \overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i)u = (\overset{h}{\nabla}{}^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j\overset{h}{\nabla}{}^i)u + (\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i)u + (A^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j A^i)u + (A^i A^j - A^j A^i)u. \quad (6.1.16)$$

Calculate the last term on the right-hand side of (6.1.16):

$$(A^i A^j - A^j A^i)u = a^{ip}\overset{v}{\nabla}_p(A^j u) - \overset{v}{\nabla}_p a^{ij} \cdot A^p u - a^{jp}\overset{v}{\nabla}_p(A^i u) + \overset{v}{\nabla}_p a^{ji} \cdot A^p u =$$

$$= a^{ip}\overset{v}{\nabla}_p(A^j u) - a^{jp}\overset{v}{\nabla}_p(A^i u) = a^{ip}\overset{v}{\nabla}_p(a^{jq}\overset{v}{\nabla}_q u) - a^{jp}\overset{v}{\nabla}_p(a^{iq}\overset{v}{\nabla}_q u).$$

Using commutability of $\overset{v}{\nabla}_p$ and $\overset{v}{\nabla}_q$, we obtain

$$(A^i A^j - A^j A^i)u = (a^{ip}\overset{v}{\nabla}_p a^{jq} - a^{jp}\overset{v}{\nabla}_p a^{iq})\overset{v}{\nabla}_q u. \qquad (6.1.17)$$

We now calculate the second term on the right-hand side of (6.1.16):

$$(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i)u = \overset{h}{\nabla}{}^i(a^{jp}\overset{v}{\nabla}_p u) - a^{jp}\overset{v}{\nabla}_p\overset{h}{\nabla}{}^i u + \overset{v}{\nabla}_p a^{ij} \cdot \overset{h}{\nabla}{}^p u.$$

Using commutability of $\overset{h}{\nabla}{}^i$ and $\overset{v}{\nabla}_p$, we infer

$$(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i)u = \overset{h}{\nabla}{}^i a^{jp} \cdot \overset{v}{\nabla}_p u + \overset{v}{\nabla}_p a^{ij} \cdot \overset{h}{\nabla}{}^p u.$$

Alternating the preceding equality with respect to i and j, we conclude that

$$(\overset{h}{\nabla}{}^i A^j - A^j\overset{h}{\nabla}{}^i)u + (A^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j A^i)u = (\overset{h}{\nabla}{}^i a^{jp} - \overset{h}{\nabla}{}^j a^{ip})\overset{v}{\nabla}_p u. \qquad (6.1.18)$$

The commutation formula (2.6.12) for $\overset{h}{\nabla}{}^i$ and $\overset{h}{\nabla}{}^j$ looks like

$$(\overset{h}{\nabla}{}^i\overset{h}{\nabla}{}^j - \overset{h}{\nabla}{}^j\overset{h}{\nabla}{}^i)u = -R^{pqij}\xi_q\overset{v}{\nabla}_p u$$

in the case of a scalar function. Inserting (6.1.17), (6.1.18) and the last expression into (6.1.16), we obtain

$$(\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j - \overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i)u = -(R^{pqij}\xi_q + \overset{h}{\nabla}{}^j a^{ip} - \overset{h}{\nabla}{}^i a^{jp} + a^{jq}\overset{v}{\nabla}_q a^{ip} - a^{iq}\overset{v}{\nabla}_q a^{jp})\overset{v}{\nabla}_p u. \qquad (6.1.19)$$

We introduce the semibasic tensor field

$$\overset{a}{R}_{ijkl} = R_{ijkl} + \overset{h}{\nabla}_l\overset{v}{\nabla}_j a_{ik} - \overset{h}{\nabla}_k\overset{v}{\nabla}_j a_{il} + a_{lp}\overset{v}{\nabla}{}^p\overset{v}{\nabla}_j a_{ik} - a_{kp}\overset{v}{\nabla}{}^p\overset{v}{\nabla}_j a_{il} + \overset{v}{\nabla}{}^p a_{ik} \cdot \overset{v}{\nabla}_j a_{lp} - \overset{v}{\nabla}{}^p a_{il} \cdot \overset{v}{\nabla}_j a_{kp}. \quad (6.1.20)$$

Contracting this equality with ξ^j and taking homogeneity (6.1.5) into account, we obtain

$$\overset{a}{R}_{ijkl}\xi^j = R_{ijkl}\xi^j + \overset{h}{\nabla}_l a_{ik} - \overset{h}{\nabla}_k a_{il} + a_{lp}\overset{v}{\nabla}^p a_{ik} - a_{kp}\overset{v}{\nabla}^p a_{il}. \tag{6.1.21}$$

In view of (6.1.21), formula (6.1.19) takes the final form:

$$(\overset{a}{\nabla}{}^i\overset{a}{\nabla}{}^j - \overset{a}{\nabla}{}^j\overset{a}{\nabla}{}^i)u = -\overset{a}{R}{}^{pqij}\xi_q\overset{v}{\nabla}_p u. \tag{6.1.22}$$

Similar but more bulky calculation yields the following commutation formula for a semibasic tensor field of arbitrary degree:

$$(\overset{a}{\nabla}{}^k\overset{a}{\nabla}{}^l - \overset{a}{\nabla}{}^l\overset{a}{\nabla}{}^k)u^{i_1\ldots i_r}_{j_1\ldots j_s} = -\overset{a}{R}{}^{pqkl}\xi_q\overset{v}{\nabla}_p u^{i_1\ldots i_r}_{j_1\ldots j_s}+$$

$$+\sum_{\alpha=1}^{r} g_{pq}\overset{a}{R}{}^{i_\alpha qkl}u^{i_1\ldots i_{\alpha-1}pi_{\alpha+1}\ldots i_r}_{j_1\ldots j_s} - \sum_{\alpha=1}^{s} g_{j_\alpha q}\overset{a}{R}{}^{pqkl}u^{i_1\ldots i_r}_{j_1\ldots j_{\alpha-1}pj_{\alpha+1}\ldots j_s}. \tag{6.1.23}$$

The semibasic tensor field $(\overset{a}{R}_{ijkl})$ defined by (6.1.20) will be referred to as the *curvature tensor for the modified horizontal derivative* $\overset{a}{\nabla}$. In what follows we need some properties of this tensor. As is seen from (6.1.20), the tensor is skew-symmetric with respect to the indices k and l but, in general, it is not skew-symmetric with respect to i and j in contrast to the conventional curvature tensor. We shall need the following properties of the tensor:

$$\overset{a}{R}_{ipkq}\xi^p\xi^q = R_{ipkq}\xi^p\xi^q + \xi^p\overset{h}{\nabla}_p a_{ik} + a_{ip}a^p_k = \overset{a}{R}_{piqk}\xi^p\xi^q, \tag{6.1.24}$$

$$\overset{a}{R}_{ipkq}\xi^p\xi^q = \overset{a}{R}_{kpiq}\xi^p\xi^q, \quad \overset{a}{R}_{pqir}\xi^p\xi^q\xi^r = \overset{a}{R}_{ipqr}\xi^p\xi^q\xi^r = 0. \tag{6.1.25}$$

Relations (6.1.25) follow from (6.1.24) on using (6.1.6). To prove the first of the equalities (6.1.24), we multiply (6.1.21) by ξ^l and sum over l. Using (6.1.6), we obtain

$$\overset{a}{R}_{ijkl}\xi^j\xi^l = R_{ijkl}\xi^j\xi^l + \xi^l\overset{h}{\nabla}_l a_{ik} - a_{kp}\xi^l\overset{v}{\nabla}^p a_{il}. \tag{6.1.26}$$

It follows from (6.1.6) that

$$\xi^i\overset{v}{\nabla}_j a_{ik} = -a_{jk}. \tag{6.1.27}$$

Transforming the last summand on the right-hand side of (6.1.26) with the help of (6.1.27), we obtain the first of the equalities (6.1.24). To prove the second, we take the contraction of (6.1.20) with $\xi^i\xi^k$:

$$\overset{a}{R}_{ijkl}\xi^i\xi^k = R_{ijkl}\xi^i\xi^k + \overset{h}{\nabla}_l(\xi^i\xi^k\overset{v}{\nabla}_j a_{ik}) - \xi^k\overset{h}{\nabla}_k(\xi^i\overset{v}{\nabla}_j a_{il})+$$

$$+a_{lp}\xi^i\xi^k\overset{v}{\nabla}{}^p\overset{v}{\nabla}_j a_{ik} + \xi^i\xi^k\overset{v}{\nabla}{}^p a_{ik}\cdot\overset{v}{\nabla}_j a_{lp} - \xi^i\overset{v}{\nabla}{}^p a_{il}\cdot\xi^k\overset{v}{\nabla}_j a_{kp}.$$

Transforming each summand on the right-hand side of the formula with the help of (6.1.27), we arrive at the second of the equalities (6.1.24).

Since the above-obtained properties of the operator $\overset{a}{\nabla}$ are similar to the corresponding properties of $\overset{h}{\nabla}$, we can assert that the following version of the Pestov identity (4.1.2) is valid for $u \in C^\infty(TM)$:

$$2\langle\overset{a}{\nabla}u, \overset{v}{\nabla}(Hu)\rangle = |\overset{a}{\nabla}u|^2 + \overset{a}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i - \overset{a}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u \tag{6.1.28}$$

with

$$v_i = \xi_i\overset{a}{\nabla}{}^j u\cdot\overset{v}{\nabla}_j u - \xi_j\overset{v}{\nabla}_i u\cdot\overset{a}{\nabla}{}^j u, \tag{6.1.29}$$

$$w^i = \xi_j\overset{a}{\nabla}{}^i u\cdot\overset{a}{\nabla}{}^j u. \tag{6.1.30}$$

Concluding the section, we will obtain a Gauss-Ostrogradskiĭ-type formula for the modified horizontal divergence. Let $u \in C^\infty(\beta^0_1 M)$ be a semibasic covector field. By the definition of (6.1.8)–(6.1.10), we have

$$\overset{a}{\nabla}{}^i u_i = \overset{h}{\nabla}{}^i u_i + A^i u_i = \overset{h}{\nabla}{}^i u_i + a^{ip}\overset{v}{\nabla}_p u_i + \overset{v}{\nabla}_i a^{ip}\cdot u_p = \overset{h}{\nabla}{}^i u_i + \overset{v}{\nabla}_p(a^{ip}u_i). \tag{6.1.31}$$

Assume the field u positively homogeneous in its second argument:

$$u(x, t\xi) = t^\lambda u(x, \xi) \quad (t > 0).$$

Multiply equality (6.1.31) by the volume form $d\Sigma^{2n-1}$ of the manifold ΩM, integrate the result over ΩM, and transform the right-hand side of the so-obtained equality by the Gauss-Ostrogradskiĭ formulas for the horizontal and vertical divergences. As a result, we obtain

$$\int\limits_{\Omega M} \overset{a}{\nabla}{}^i u_i \, d\Sigma^{2n-1} = \int\limits_{\partial\Omega M} \nu^i u_i \, d\Sigma^{2n-2} + (\lambda + n - 1) \int\limits_{\Omega M} \xi_p a^{ip} u_i \, d\Sigma^{2n-1}.$$

Observing that the integrand of the second integral on the right-hand side equals zero by (6.1.6), we arrive at the following Gauss-Ostrogradskiĭ formula:

$$\int\limits_{\Omega M} \overset{a}{\nabla}{}^i u_i \, d\Sigma^{2n-1} = \int\limits_{\partial\Omega M} \langle \nu, u \rangle \, d\Sigma^{2n-2}. \tag{6.1.32}$$

6.2 Constructing the modifying tensor field

We say that a linear system $d^2y/dt^2 + A(t)y = 0$ $(y = (y_1, \ldots, y_n))$ of second order differential equations has no cojugate points on a segment $[a, b]$ if there is no nontrivial solution to the system which vanishes at two different points of the segment.

Look at the last summand on the right-hand side of Pestov's identity (6.1.28). Since in applications of Pestov's identity some additional terms may appear of the same kind as the last term on the right-hand side of (6.1.28), we formulate our result as follows:

Theorem 6.2.1 *Let (M, g) be a CDRM, and let $S \in C^\infty(\beta_4^0 M; T^0 M)$ be a semibasic tensor field on $T^0 M$ possessing the properties*

$$S_{ipjq}\xi^p\xi^q = S_{jpiq}\xi^p\xi^q = S_{piqj}\xi^p\xi^q, \quad S_{ipqr}\xi^p\xi^q\xi^r = 0$$

and positively homogeneous of degree zero in ξ:

$$S_{ijkl}(x, \lambda\xi) = S_{ijkl}(x, \xi) \quad (\lambda > 0).$$

Assume that, for every $(x, \xi) \in \Omega M$, the equation

$$\frac{D^2\eta}{dt^2} + (R + S)(t)\eta = 0 \tag{6.2.1}$$

lacks conjugate points on the geodesic $\gamma = \gamma_{x,\xi} : [\tau_-(x, \xi), \tau_+(x, \xi)] \to M$ (here $[(R + S)(t)]_k^p = g^{ip}(R_{ijkl} + S_{ijkl})(\gamma(t), \dot\gamma(t))\dot\gamma^j(t)\dot\gamma^l(t)$). Then there exists a semibasic tensor field $a = (a^{ij}) \in C^\infty(\beta_0^2 M; T^0 M)$ on $T^0 M$ satisfying (6.1.4)–(6.1.6) and such that the corresponding curvature tensor $\overset{a}{R}$ defined by formula (6.1.20) meets the equation

$$(\overset{a}{R}_{ijkl} + S_{ijkl})\xi^i\xi^k = 0. \tag{6.2.2}$$

Proof. By (6.1.24), equation (6.2.2) is equivalent to the following:

$$(Ha)_{ij} + a_{ip}a_j^p + \widetilde{R}_{ipjq}\xi^p\xi^q = 0, \tag{6.2.3}$$

where $\widetilde{R} = R + S$.

If the field $a(x, \xi)$ is positively homogeneous of degree 1 in ξ, then the left-hand side of equation (6.2.3) is positively homogeneous of degree 2. Conversely, with a solution to equation (6.1.35) on ΩM available, we obtain a solution on the whole of $T^0 M$, using extension by homogeneity. Therefore, we further consider equation (6.2.3) on ΩM.

We represent ΩM as the union of disjoint one-dimensional submanifolds, the orbits of the geodesic flow. Restricted to an orbit, (6.2.3) gives a system of ordinary differential equations. For distinct orbits, the systems do not relate to one another. Having the equation solved on each orbit, we must then take care that the family of solutions forms a smooth field on the whole of ΩM. This can be achieved by appropriately choosing the initial values on the orbits. We proceed to implementing the plan.

Given $(x, \xi) \in \partial_-\Omega M$, we consider a maximal geodesic $\gamma = \gamma_{x,\xi} : [0, \tau_+(x, \xi)] \to M$ satisfying the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. Taking $x = \gamma(t)$ and $\xi = \dot\gamma(t)$ in (6.2.3), we obtain the system of ordinary differential equations of Riccati type:

$$\left(\frac{Da}{dt}\right)_{ij} + a_{ip}a_j^p + \widetilde{R}_{ipjq}\dot\gamma^p\dot\gamma^q = 0. \tag{6.2.4}$$

To prove the theorem, it suffices to establish existence of a symmetric solution $(a_{ij}(t))$ to system (6.2.4) on the interval $[0, \tau_+(x, \xi)]$, the solution dependent smoothly on $(x, \xi) \in \partial_-\Omega M$ and satisfying the additional condition

$$a_{ij}(t)\dot{\gamma}^j(t) = 0. \tag{6.2.5}$$

Contracting (6.2.4) with $\dot{\gamma}^j$, we see that an arbitrary solution to system (6.2.4) meets (6.2.5), provided that the condition is satisfied at $t = 0$. Demonstrate that a similar assertion is valid as regards the symmetry of a_{ij}. Indeed, an arbitrary solution $a_{ij}(t)$ to system (6.2.4) is representable as $a_{ij} = a_{ij}^+ + a_{ij}^-$ where a_{ij}^+ is symmetric and a_{ij}^- is skew-symmetric. Inserting this expression into (6.2.4), we obtain

$$\left[\left(\frac{Da^+}{dt}\right)_{ij} + g^{pq}\left(a_{ip}^+ a_{qj}^+ + a_{ip}^- a_{qj}^-\right) + \widetilde{R}_{ipjq}\dot{\gamma}^p\dot{\gamma}^q\right] + \left[\left(\frac{Da^-}{dt}\right)_{ij} + g^{pq}\left(a_{ip}^+ a_{qj}^- + a_{ip}^- a_{qj}^+\right)\right] = 0.$$

The expression in the first brackets is symmetric and that in the second is skew-symmetric. Consequently,

$$\left(\frac{Da^-}{dt}\right)_{ij} + g^{pq}\left(a_{ip}^+ a_{qj}^- + a_{ip}^- a_{qj}^+\right) = 0.$$

The last equalities can be considered as a homogeneous linear system in a_{ij}^-. The system, together with the initial condition $a^-(0) = 0$, implies that $a^- \equiv 0$.

Thus, symmetry of the field (a_{ij}) and its orthogonality to the vector ξ are insured by the choice of the initial value. We now consider the question of existence of a solution to system (6.2.4). Raising the index i, we rewrite the system as

$$\left(\frac{Da}{dt}\right)^i_j + a^i_p a^p_j + \widetilde{R}^i_j = 0 \quad (\widetilde{R}^i_j = \widetilde{R}^i{}_{pjq}\dot{\gamma}^p\dot{\gamma}^q)$$

or, in matrix form, as

$$\frac{Da}{dt} + a^2 + \widetilde{R} = 0. \tag{6.2.6}$$

We look for a solution to this equation in the form

$$a = \frac{Db}{dt}b^{-1}. \tag{6.2.7}$$

Inserting (6.2.7) into (6.2.6), we arrive at the equation

$$\frac{D^2 b}{dt^2} + \widetilde{R}b = 0. \tag{6.2.8}$$

Conversely, if equation (6.2.8) has a nondegenerate solution b, then equation (6.2.6) is satisfied by the matrix a defined by formula (6.2.7).

We denote by $z = (z^i_j(x, \xi; t))$ and $w = (w^i_j(x, \xi; t))$ solutions to equation (6.2.8) satisfying the initial conditions

$$z(0) = 0, \quad \left(\frac{Dz}{dt}(0)\right)^i_j = \delta^i_j; \qquad w^i_j(0) = \delta^i_j, \quad \frac{Dw}{dt}(0) = 0. \tag{6.2.9}$$

Observe that the fields $z(x, \xi; t)$ and $w(x, \xi; t)$ are smooth in all of their arguments. By the condition of the theorem stipulating the absence of conjugate points, the matrix $z^i_j(x, \xi; t)$ is nondegenerate for $0 < t \leq \tau_+(x, \xi)$. By initial conditions (6.2.9), there is a $t_0 > 0$ such that the matrices $z(x, \xi; t)$ and $w(x, \xi; t)$ are positive definite for $0 < t \leq \tau(x, \xi) = \min(t_0, \tau_+(x, \xi))$. Consequently, the matrix

$$\tilde{b}(x, \xi; t) = \lambda z + w \tag{6.2.10}$$

is nondegenerate for $0 \leq t \leq \tau(x, \xi)$ and every $\lambda > 0$. The determinant of the matrix $z(x, \xi; t)$ is bounded from below by some positive constant uniformly in $(x, \xi) \in \partial_-\Omega M$ and $t_0 \leq t \leq \tau_+(x, \xi)$. Therefore, choosing a sufficiently large positive constant λ in (6.2.10), we can guarantee that the matrix $\tilde{b}(x, \xi; t)$ is nondegenerate for all $(x, \xi; t)$ in the set

$$G = \{(x, \xi; t) \mid (x, \xi) \in \partial_-\Omega M, \ 0 \leq t \leq \tau_+(x, \xi)\}.$$

Thus, we have found a nondegenerate solution $\tilde{b} = (\tilde{b}^i_j(x, \xi; t))$ to equation (6.2.8) depending smoothly on $(x, \xi; t) \in G$ and satisfying the initial conditions

$$\tilde{b}^i_j(0) = \delta^i_j, \quad \left(\frac{D\tilde{b}}{dt}(0)\right)^i_j = \lambda \delta^i_j. \tag{6.2.11}$$

We now assign

$$b^i_j(x, \xi; t) = \tilde{b}^i_j(x, \xi; t) - \lambda t \dot{\gamma}^i_{x,\xi}(t) \dot{\gamma}_{j,x,\xi}(t) \quad ((x, \xi; t) \in G). \tag{6.2.12}$$

The matrix $b = (b^i_j)$ meets (6.2.8) and the initial conditions

$$b^i_j(0) = \delta^i_j, \quad \left(\frac{Db}{dt}(0)\right)^i_j = (\delta^i_j - \xi^i \xi_j)\lambda. \tag{6.2.13}$$

Demonstrate that the matrix $b(x, \xi; t)$ is nondegenerate for all $(x, \xi; t) \in G$. Indeed, let $\gamma = \gamma_{x,\xi}$ and $0 \neq \eta \in T_{\gamma(t)}M$. Represent η as $\eta = \tilde{\eta} + \mu \dot{\gamma}(t)$, where $\tilde{\eta} \perp \dot{\gamma}(t)$ and $|\tilde{\eta}|^2 + \mu^2 > 0$. Then

$$b^i_j(t)\eta^j = (\tilde{b}^i_j - \lambda t \dot{\gamma}^i \dot{\gamma}_j)(\tilde{\eta}^j + \mu \dot{\gamma}^j) = \tilde{b}^i_j \tilde{\eta}^j + \mu(\tilde{b}^i_j \dot{\gamma}^j - \lambda t \dot{\gamma}^i). \tag{6.2.14}$$

Since $\tilde{b}(t)$ satisfies equation (6.2.8) and initial condition (6.2.11), we have

$$\tilde{b}^i_j(t)\dot{\gamma}^j(t) = (1 + \lambda t)\dot{\gamma}^i(t).$$

In view of the last equality, (6.2.14) implies

$$b^i_j(t)\eta^j = \tilde{b}^i_j(t)\left[\tilde{\eta}^j + \frac{\mu}{1 + \lambda t}\dot{\gamma}^j(t)\right]. \tag{6.2.15}$$

The vector in the brackets is nonzero, since $\lambda > 0$, $t \geq 0$ and $\tilde{\eta} \perp \dot{\gamma}(t)$. Since the matrix $(\tilde{b}^i_j(t))$ is nondegenerate, the right-hand side of equality (6.2.15) differs from zero for every $\eta \neq 0$. Since this is true for each t, the matrix $b = (b^i_j(x, \xi; t))$ is nondegenerate.

We have thus constructed a nondegenerate solution $b = (b^i_j(x, \xi; t))$ to equation (6.2.8) which depends smoothly on $(x, \xi; t) \in G$ and satisfies initial conditions (6.2.9). Consequently, the matrix $a = (a^i_j(x, \xi; t))$ defined by formula (6.2.7) satisfies equation (6.2.6) and the initial condition

$$a^i_j(x, \xi; 0) = \lambda(\delta^i_j - \xi^i \xi_j).$$

Lowering the superscript i, we obtain

$$a_{ij}(x, \xi; 0) = \lambda(g_{ij} - \xi_i \xi_j).$$

Whence we see that the tensor $a_{ij}(x, \xi; 0)$ is symmetric and orthogonal to the vector ξ. As mentioned, validity of these properties at $t = 0$ implies their validity for all t. The theorem is proved.

6.3 Finiteness theorem for the ray transform

By Theorem 3.3.1, the ray transform on a CDRM is extendible to the bounded operator

$$I : H^k(S^m \tau'_M) \to H^k(\partial_+ \Omega M) \tag{6.3.1}$$

for every integer $k \geq 0$. We denote the kernel of this operator by $Z^k(S^m \tau'_M)$. Let us recall that a tensor field $f \in H^k(S^m \tau'_M)$ is called potential if it can be represented in the form $f = dv$ with some $v \in H^{k+1}(S^{m-1} \tau'_M)$ satisfying the boundary condition $v|_{\partial M} = 0$. Let $P^k(S^m \tau'_M)$ be the subspace, of $H^k(S^m \tau'_m)$, consisting of all potential fields. By Lemma 3.4.1, there is the inclusion

$$P^k(S^m \tau'_M) \subset Z^k(S^m \tau'_M), \tag{6.3.2}$$

Problem 3.4.2 of inverting the ray transform is equivalent to the following question: For what classes of CDRMs and for what values of k and m can the inclusion in (6.3.2) be replaced with equality?

As can be easily shown, if the answer is positive for $k = k_0$, then it is positive for $k \geq k_0$. Theorem 3.4.3 gives the positive answer for $k = 1$ and for all m under some assumption (depending on m) on the curvature of the metric.

The main result of this section is the following

Theorem 6.3.1 *Given a simple compact Riemannian manifold (M, g), inclusion $(6.3.2)$ is of a finite codimension for all m and $k \geq 1$.*

Together with the proof of Theorem 6.3.1, we shall establish the next

Theorem 6.3.2 *If (M, g) is a simple compact Riemannian manifold, then inclusion $(6.3.2)$ is the equality for $m = 0$ or $m = 1$ and for all $k \geq 1$.*

The last claim is not new; for $m = 0$ it was proved in [58, 14]; and for $m = 1$ it was proved in [10]. In conclusion of the section we formulate some problems.

Problem 6.3.3 *Does there exist a simple compact Riemannian manifold for which inclusion $(6.3.5)$ is not equality?*

To author's opinion, such manifolds exist; but the author had no success in constructing an example.

Problem 6.3.4 *Given a simple Riemannian manifold, is the codimension $c_{k,m}(M, g)$ of inclusion $(6.3.5)$ independent of k? In other words, does there exist a complement of $P^k(S^m \tau_M')$, in $Z^k(S^m \tau_M')$, consisting of smooth tensor fields?*

Problem 6.3.5 *Does there exist a CDRM for which inclusion $(6.3.5)$ is of infinite codimension?*

6.4   Proof of Theorem 6.3.1

Lemma 6.4.1 *Given a CDRM (M, g), the operator*

$$L : C(\Omega M) \to C(\Omega M)$$

defined by the equality

$$(LF)(x, \xi) = u(x, \xi) = \int\limits_{\tau_-(x,\xi)}^{0} F(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)) \, dt \qquad (6.4.1)$$

is extendible to the bounded operator

$$L : L_2(\Omega M) \to L_2(\Omega M). \qquad (6.4.2)$$

Proof. First we consider the case of $F \in C(\Omega M)$. Given by formula $(6.4.1)$, the function $u(x, \xi)$ belongs to $C(\Omega M)$. We will obtain an estimate of the norm $\|u\|_{L_2(\Omega M)}$. To this end, we transform the integral

$$\|u\|_{L_2(\Omega M)}^2 = \int\limits_{\Omega M} |u(y, \eta)|^2 \, d\Sigma^{2n-1}(y, \eta)$$

by the Santalo formula $(3.3.4)$:

$$\|u\|_{L_2(\Omega M)}^2 = \int\limits_{\partial_+ \Omega M} \langle \xi, \nu(x) \rangle \left[\int\limits_{\tau_-(x,\xi)}^{0} |u(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))|^2 \, dt \right] d\Sigma^{2n-2}(x, \xi). \qquad (6.4.3)$$

Definition $(6.4.1)$ of the function $u(y, \eta)$ can be rewritten as follows:

$$u(y, \eta) = \int\limits_{\tau_-(y,\eta)}^{0} F(\gamma_{y,\eta}(s), \dot{\gamma}_{y,\eta}(s)) \, ds.$$

Putting $(y, \eta) = (\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))$ here, we obtain

$$u(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)) = \int\limits_{\tau_-(x,\xi)-t}^{0} F(\gamma_{x,\xi}(s+t), \dot{\gamma}_{x,\xi}(s+t)) \, ds = \int\limits_{\tau_-(x,\xi)}^{t} F(\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s)) \, ds.$$

With the help of the Cauchy-Bunjakovskiĭ inequality we obtain

$$|u(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))|^2 \leq (t - \tau_-(x,\xi)) \int\limits_{\tau_-(x,\xi)}^{t} |F(\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s))|^2 \, ds.$$

The last inequality and (6.4.3) imply

$$\|u\|^2_{L_2(\Omega M)} \leq \int\limits_{\partial_+\Omega M} \langle \xi, \nu(x)\rangle \left[\int\limits_{\tau_-(x,\xi)}^{0} (t - \tau_-(x,\xi)) \, dt \int\limits_{\tau_-(x,\xi)}^{t} |F(\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s))|^2 \, ds\right] d\Sigma^{2n-2}(x,\xi).$$

After changing the integration limits t and s, this inequality takes the form

$$\|u\|^2_{L_2(\Omega M)} \leq \int\limits_{\partial_+\Omega M} \langle \xi, \nu(x)\rangle \left[\int\limits_{\tau_-(x,\xi)}^{0} (s\tau_-(x,\xi) - s^2/2)|F(\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s))|^2 \, ds\right] d\Sigma^{2n-2}(x,\xi).$$

We return to the integration variable $(y, \eta) = (\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s))$ in the last integral. Taking the relations $s = -\tau_+(y, \eta)$ and $\tau_-(x,\xi) = \tau_-(y,\eta) - \tau_+(y,\eta)$ into account, we obtain the inequality

$$\|u\|^2_{L_2(\Omega M)} \leq \int\limits_{\Omega M} \tau_+(x,\xi) \left(\frac{1}{2}\tau_+(x,\xi) - \tau_-(x,\xi)\right) |F(x,\xi)|^2 \, d\Sigma^{2n-1}(x,\xi)$$

which implies the estimate

$$\|u\|_{L_2(\Omega M)} \leq C\|F\|_{L_2(\Omega M)}.$$

Being proved for $u \in C(\Omega M)$, the last estimate allows us to finish the proof of the theorem by standard arguments.

The main step in our proof of Theorem 6.3.1 is the next

Lemma 6.4.2 *Let (M, g) be a simple compact Riemannian manifold. For every field $f \in C^\infty(S^m\tau'_M)$, the function $Lf = u \in C(\Omega M)$, defined by equality (6.4.1) with $F(x,\xi) = f_{i_1\ldots i_m}(x)\xi^{i_1}\ldots\xi^{i_m}$, belongs to $H^1(\Omega M)$ and satisfies the estimate*

$$\|u\|^2_{H^1(\Omega M)} \leq C \left[m\|u\|^2_{L_2(\Omega M)} + m\|\delta f\|_{L_2(S^{m-1}\tau'_M)} \cdot \|u\|_{L_2(\Omega M)} + \right.$$

$$\left. + m\|If\|_{L_2(\partial_+\Omega M)} \cdot \|j_\nu f|\partial M\|_{L_2(S^{m-1}\tau'_M|\partial M)} + \|If\|^2_{H^1(\partial_+\Omega M)}\right] \tag{6.4.4}$$

with some constant C independent of f.

The proof of the lemma will be given in the next section, and now we will prove Theorem 6.3.1 with use made of the lemma. First of all, Lemma 6.4.2 implies the next

Corollary 6.4.3 *Given a simple Riemannian manifold (M, g), the operator $f \mapsto u$, defined by formula (6.4.1) with $F(x,\xi) = f_{i_1\ldots i_m}(x)\xi^{i_1}\ldots\xi^{i_m}$, is extendible to the bounded operator*

$$L : H^1(S^m\tau'_M) \rightarrow H^1(\Omega M). \tag{6.4.5}$$

For $f \in H^1(S^m\tau'_M)$ and $u = Lf$, estimate (6.4.4) is valid.

Proof. Given $f \in H^1(S^m\tau'_M)$, let $f_k \in C^\infty(S^m\tau'_M)$ $(k = 1, 2, \ldots)$ be a sequence converging to f,

$$f_k \rightarrow f \quad \text{in} \quad H^1(S^m\tau'_M) \quad \text{as} \quad k \rightarrow \infty.$$

Then

$$\delta f_k \rightarrow \delta f \quad \text{in} \quad L_2(S^{m-1}\tau'_M)$$

and, by boundedness of the operator I,

$$If_k \rightarrow If \quad \text{in} \quad H^1(\partial_+\Omega M).$$

Besides, by Lemma 6.4.1,
$$Lf_k = u_k \to u = Lf \quad \text{in} \quad L_2(\Omega M). \tag{6.4.6}$$
Applying estimate (6.4.4) to the difference $u_k - u_l$, we see that u_k is a Cauchy sequence in $H^1(\Omega M)$ and, consequently, it converges in $H^1(\Omega M)$. Therefore (6.4.6) implies that $u \in H^1(\Omega M)$ and
$$u_k \to u \quad \text{in} \quad H^1(\Omega M).$$

Writing down estimate (6.4.4) for u_k and passing to the limit as $k \to \infty$ in this inequality, we arrive at estimate (6.4.4) for u.

Proof of Theorem 6.3.1. First of all we show that the claim of the theorem for $k = 1$ implies the same for arbitrary $k \geq 1$.

The kernel $Z^k(S^m\tau_M')$ is the closed subspace in the Hilbert space $H^k(S^m\tau_M')$. Let
$$A^{k,m} = Z^k(S^m\tau_M') \ominus P^k(S^m\tau_M')$$
be the orthogonal complement of the space of potential fields in $Z^k(S^m\tau_M')$ with respect to the scalar product
$$(u,v)_{L_2(\Omega M)} = \int\limits_{\Omega M} \langle u(x,\xi), v(x,\xi) \rangle \, d\Sigma(x,\xi).$$

The claim of Theorem 6.5.1 is equivalent to finiteness of dimension of $A^{k,m}$. It follows from the Green formula (2.4.2) for d and δ that $A^{k,m}$ consists of all fields $f \in H^k(S^m\tau_M')$ satisfying the relations
$$\delta f = 0, \qquad If = 0. \tag{6.4.7}$$
Consequently, $A^{k,m} \subset A^{k',m}$ for $k \geq k'$. Thus, in what follows we consider the case of $k = 1$.

We have to prove that the space $A^{1,m}$ has a finite dimension. To this end we consider the image $L(A^{1,m})$ of the space with respect to the operator L defined by (6.4.2). Note that the operator L is injective. Indeed, as we know, the function $u = Lf$ satisfies equation (3.5.4) that recovers f from u. Therefore to prove the theorem it suffices to show that the subspace $L(A^{1,m}) \subset H^1(\Omega M)$ has a finite dimension.

For $f \in A^{1,m}$ and $u = Lf$, estimate (6.4.4) is valid. By (6.4.7), the estimate takes the form
$$\|u\|_{H^1(\Omega M)} \leq Cm\|u\|_{L_2(\Omega M)}. \tag{6.4.8}$$
Thus, estimate (6.4.8) holds for every $u \in L(A^{1,m})$. Since the imbedding $H^1(\Omega M) \subset L_2(\Omega M)$ is compact, estimate (6.4.8) implies finiteness of the dimension of $L(A^{1,m})$. The theorem is proved.

In the case of $m = 0$, estimate (6.4.8) gives us $u \equiv 0$. Therefore $A^{1,0} = 0$ that is equivalent to the claim of Theorem 6.3.2 in the case of $m = 0$.

6.5   Proof of Lemma 6.4.2

Before proving Lemma 6.4.2 we will establish some auxiliary claims.

Lemma 6.5.1 *Let (M,g) be a CDRM, and $\lambda \geq 0$ be a continuous function on ΩM. Assume a non-negative function $\varphi \in C(\Omega M)$ to be smooth on $\Omega_\varphi = \{(x,\xi) \in \Omega M \mid \varphi(x,\xi) > 0\}$, satisfy the boundary condition*
$$\varphi|_{\partial_-\Omega M} = 0$$
and the next condition
$$\sup_{(x,\xi) \in \Omega_\varphi} |H\varphi(x,\xi)| < \infty.$$
Then the estimate
$$\int\limits_{\Omega M} \lambda|\varphi|^2 \, d\Sigma \leq C \int\limits_{\Omega_\varphi} [H\varphi]_+^2 \, d\Sigma$$
holds with some constant C independent of φ; here the notation
$$[a]_+ = \begin{cases} a, & \text{if } a \geq 0 \\ 0, & \text{if } a < 0 \end{cases}$$
is used.

The proof of this claim can be obtained by a slight modification of the proof of Lemma 4.2.2. Namely, using nonnegativity of φ, formula (4.2.15) implies the inequality

$$|\psi_y(t)| \leq \int\limits_0^t \left[\frac{d\psi_y(\tau)}{d\tau} \right]_+ d\tau.$$

The rest of arguments is not changed.

Lemma 6.5.2 *Let (M, g) be a CDRM, and $a \in C^\infty(\beta_1^1 M)$. By $A : C^\infty(\beta_m^0 M) \to C^\infty(\beta_m^0 M)$ we denote the differential operator defined in coordinate form by the equality*

$$(Au)_{i_1 \ldots i_m} = (Hu)_{i_1 \ldots i_m} + a_{i_1}^j u_{j i_2 \ldots i_m}. \tag{6.5.1}$$

If a field $u \in C^\infty(\beta_m^0 M)$ satisfies the boundary condition

$$u|_{\partial_- \Omega M} = 0, \tag{6.5.2}$$

then the estimate

$$\|u\|_{L_2(\Omega M)} \leq C \|Au\|_{L_2(\Omega M)} \tag{6.5.3}$$

holds with some constant C independent of u.

Proof. The function $\tilde\varphi = |u|$ is continuous on ΩM, smooth on $\Omega_{\tilde\varphi} = \{(x, \xi) \in \Omega M \mid \tilde\varphi(x, \xi) > 0\}$ and satisfies the boundary condition $\tilde\varphi|_{\partial_- \Omega M} = 0$. The equality $H\tilde\varphi = \langle u, Hu \rangle / |u|$ holds on $\Omega_{\tilde\varphi}$ and, consequently,

$$|H\tilde\varphi| \leq |Hu|. \tag{6.5.4}$$

From (6.5.1) we obtain the relation

$$\frac{1}{2} H(|u|^2) = \langle Au, u \rangle - \langle au, u \rangle$$

which implies the inequality

$$|u| \cdot H(|u|) \leq |Au| \cdot |u| + |a| \cdot |u|^2.$$

With the help of (6.5.4), it implies that the inequality

$$H\tilde\varphi - |a|\tilde\varphi \leq |Au|$$

holds on $\Omega_{\tilde\varphi}$. It can be rewritten in the form

$$H(e^{-b}\tilde\varphi) \leq e^{-b}|Au|, \tag{6.5.5}$$

where b is a function on ΩM satisfying the equation $Hb = |a|$.

The function $\varphi = e^{-b}\tilde\varphi$ satisfies the conditions of Lemma 6.5.1. Applying this lemma with $\lambda = e^{2b}$ and using (6.5.5), we obtain

$$\|u\|_{L_2(\Omega M)}^2 = \int\limits_{\Omega M} |\tilde\varphi|^2 \, d\Sigma = \int\limits_{\Omega M} \lambda |\varphi|^2 \, d\Sigma \leq C \int\limits_{\Omega_\varphi} [H\varphi]_+^2 \, d\Sigma \leq C_1 \int\limits_{\Omega M} |Au|^2 \, d\Sigma = C_1 \|Au\|_{L_2(\Omega M)}^2.$$

The lemma is proved.

Proof of Lemma 6.4.2. Let $f \in C^\infty(S^m \tau_M')$. In what follows we agree to denote various constants independent of f by the same letter C.

Given f, we define the function $u \in C(T^0 M)$ by formula (3.5.1). This function is smooth on $T^0 M \setminus T(\partial M)$, satisfies the kinetic equation (3.5.4), the boundary conditions (3.5.2)–(3.5.3), and the homogeneity condition (3.5.6).

Applying Theorem 6.2.1 with $S = 0$, we can find a modifying tensor field $a \in C^\infty(\beta_0^2 M; T^0 M)$ on $T^0 M$ such that the curvature tensor of the corresponding modified horizontal derivative $\overset{a}{\nabla}$ satisfies the equation

$$\overset{a}{R}_{ijkl} \xi^i \xi^k = 0. \tag{6.5.6}$$

Note that it is the unique point in our proof where simplicity of (M, g) is used. The conditions of Theorem 6.2.1 require absence of conjugate points for the Jacobi equation; that is equivalent to simplicity of a CDRM.

We write down the Pestov identity (6.1.28) for the function u:

$$2\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = |\overset{a}{\nabla} u|^2 + \overset{a}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i. \tag{6.5.7}$$

The term containing the curvature vanishes because of (6.5.6). The semibasic fields (v_i) and (w^i) are defined by the formulas (6.1.29) and (6.1.30).

We transform the left-hand side of (6.5.7). By (3.5.4),

$$\overset{v}{\nabla}_i(Hu) = \overset{v}{\nabla}_i(f_{i_1\dots i_m}\xi^{i_1}\dots\xi^{i_m}) = m f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}. \tag{6.5.8}$$

Therefore

$$2\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = 2m\overset{a}{\nabla}{}^i u \cdot f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} = \overset{a}{\nabla}{}^i(2mu f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) - 2mu\overset{a}{\nabla}{}^i(f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}).$$

Introducing the notation

$$\tilde{v}^i = 2mu f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}, \tag{6.5.9}$$

we obtain

$$2\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = \overset{a}{\nabla}{}^i\tilde{v}_i - 2mu\overset{a}{\nabla}{}^i(f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}).$$

Using the definition of the modified derivative, the last formula is transformed as follows:

$$2\langle \overset{a}{\nabla} u, \overset{v}{\nabla}(Hu)\rangle = \overset{a}{\nabla}{}^i\tilde{v}_i - 2mu\left[\overset{h}{\nabla}{}^i(f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) + a^{ij}\overset{v}{\nabla}_j(f_{i i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) + \overset{v}{\nabla}_i a^{ip}\cdot f_{p i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}\right]$$

$$= \overset{a}{\nabla}{}^i\tilde{v}_i - 2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} - 2m(m-1)a^{ij}u f_{i j i_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} - 2m\overset{v}{\nabla}_i a^{ip}\cdot u f_{p i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.$$

Inserting this expression into (6.5.7), we obtain

$$|\overset{a}{\nabla} u|^2 = -2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} -$$

$$-2m\overset{v}{\nabla}_i a^{ip}\cdot u f_{p i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} - 2m(m-1)a^{ij}u f_{i j i_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} + \overset{a}{\nabla}{}^i(\tilde{v}_i - v_i) - \overset{v}{\nabla}_i w^i. \tag{6.5.10}$$

The function u depends smoothly on $(x, \xi) \in T^0 M$ except for the points of the set $T^0(\partial M)$ where some derivatives of u can be infinite. Consequently, some of the integrals considered below are improper and we have to verify their convergence. The verification is performed in the same way as in Section 4.3, since the singularities of u are due only to the singularities of the lower integration limit in (3.5.1). So we will not pay attention to these singularities in what follows.

We multiply equality (6.5.10) by the volume form $d\Sigma = d\Sigma^{2n-1}$ and integrate it over ΩM. Transforming the integrals of divergent terms by the Gauss-Ostrogradskiĭ formulas (Theorem 2.7.1 and formula (6.1.32), we obtain

$$\int\limits_{\Omega M} |\overset{a}{\nabla} u|^2\, d\Sigma = \int\limits_{\partial\Omega M} \langle \tilde{v} - v, \nu\rangle\, d\Sigma^{2n-2} - (n+2m-2)\int\limits_{\Omega M} \langle w, \xi\rangle\, d\Sigma -$$

$$- 2m\int\limits_{\Omega M}\left[u(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} + (m-1)a^{ij}u f_{i j i_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} - \overset{v}{\nabla}_i a^{ip}\cdot u f_{p i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}\right] d\Sigma.$$

By (6.1.30), $\langle w, \xi\rangle = |Hu|^2$ and the previous formula takes the form

$$\int\limits_{\Omega M}\left(|\overset{a}{\nabla} u|^2 + (n+2m-2)|Hu|^2\right) d\Sigma = \int\limits_{\partial\Omega M} \langle \tilde{v} - v, \nu\rangle\, d\Sigma^{2n-2} -$$

$$- 2m\int\limits_{\Omega M}\left[u(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} + (m-1)a^{ij}u f_{i j i_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} + \overset{v}{\nabla}_i a^{ip}\cdot u f_{p i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}\right] d\Sigma.$$

$$\tag{6.5.11}$$

Repeating the arguments of the end of Section 4.3, we insure that the first integral on the right-hand side of (6.5.11) can be estimated as follows:

$$\left| \int_{\partial \Omega M} \langle \tilde{v} - v, \nu \rangle \, d\Sigma^{2n-2} \right| \leq C N^2(f) \equiv C(m\|j_\nu f|_{\partial M}\|_0 \cdot \|If\|_0 + \|If\|_1^2). \qquad (6.5.12)$$

Hereafter we use the brief notations for norms:

$$\|If\|_k = \|If\|_{H^k(\partial_+ \Omega M)}, \quad \|f\|_k = \|f\|_{H^k(S^m \tau'_M)}, \quad \|u\|_k = \|u\|_{H^k(\Omega M)}$$

and so on.

By Lemma 6.4.1, $u \in L_2(\Omega M)$. From this with the help of (6.5.11) and (6.5.12) we obtain that $\overset{a}{\nabla} u \in L_2(\beta_0^1 M; \Omega M)$ and $Hu \in L_2(\Omega M)$ as well as the inequality

$$\|\overset{a}{\nabla} u\|_0^2 + \|Hu\|_0^2 \leq C \left(m\|u\|_0 \cdot \|\delta f\|_0 + m\|u\|_0 \cdot \|f\|_0 + N^2(f) \right). \qquad (6.5.13)$$

Besides this, the kinetic equation (3.5.4) implies the estimate

$$\|f\|_0 \leq C\|Hu\|_0. \qquad (6.5.14)$$

It follows from (6.5.13) and (6.5.14) that

$$\|Hu\|_0^2 \leq C \left(\|u\|_0 \cdot \|Hu\|_0 + \|u\|_0 \cdot \|\delta f\|_0 + N^2(f) \right). \qquad (6.5.15)$$

Considering (6.5.15) as a square inequality in $\|Hu\|_0$, we obtain

$$\|Hu\|_0 \leq C \left(\|u\|_0 + \|\delta f\|_0 + N(f) \right). \qquad (6.5.16)$$

The estimates (6.5.14) and (6.5.16) imply the inequality

$$\|f\|_0 \leq C \left(\|u\|_0 + \|\delta f\|_0 + N(f) \right)$$

with help of which (6.5.13) gives

$$\|\overset{a}{\nabla} u\|_0^2 \leq C \left(m\|u\|_0^2 + m\|u\|_0 \cdot \|\delta f\|_0 + N^2(f) \right). \qquad (6.5.17)$$

We now estimate $\|\overset{v}{\nabla} u\|_0$ by $\|\overset{a}{\nabla} u\|_0$. From (3.5.4) with the help of the commutation formula $\overset{v}{\nabla} H - H \overset{v}{\nabla} = \overset{h}{\nabla}$ we obtain

$$H \overset{v}{\nabla}_i u = -\overset{h}{\nabla}_i u + m f_{ii_2 \ldots i_m} \xi^{i_2} \ldots \xi^{i_m}. \qquad (6.5.18)$$

By the definition of the modified derivative

$$\overset{h}{\nabla}_i u = g_{ij} \overset{a}{\nabla}^j u - a_i^j \overset{v}{\nabla}_j u.$$

Inserting this expression into (6.5.18), we obtain

$$(A \overset{v}{\nabla} u)_i = (H \overset{v}{\nabla} u)_i - a_i^j \overset{v}{\nabla}_j u = -g_{ij} \overset{a}{\nabla}^j u + m f_{ii_2 \ldots i_m} \xi^{i_2} \ldots \xi^{i_m}. \qquad (6.5.19)$$

By (3.5.2), the field $\overset{v}{\nabla} u$ satisfies the boundary condition $\overset{v}{\nabla} u|_{\partial_- \Omega M} = 0$. Applying Lemma 6.5.2 to the field $\overset{v}{\nabla} u$ and operator A defined by (6.5.19), we arrive at the estimate

$$\|\overset{v}{\nabla} u\|_0^2 \leq C(\|\overset{a}{\nabla} u\|_0^2 + \|f\|_0^2). \qquad (6.5.20)$$

The equality $Hu = \xi_i \overset{a}{\nabla}^i u$ and estimate (6.5.14) imply the inequality $\|f\|_0 \leq C\|\overset{a}{\nabla} u\|$. With the help of the latter, (6.5.20) gives

$$\|\overset{v}{\nabla} u\|_0 \leq C\|\overset{a}{\nabla} u\|_0. \qquad (6.5.21)$$

Finally, the estimate

$$\|u\|_0 \leq C\|Hu\|_0 \leq C_1\|\overset{a}{\nabla} u\|_0 \qquad (6.5.22)$$

is obtained by applying Lemma 6.5.2 with $A = H$.

The next three norms

$$\|u\|_{H^1(\Omega M)}, \quad (\|\overset{h}{\nabla}u\|_0^2 + \|\overset{v}{\nabla}u\|_0^2 + \|u\|_0^2)^{1/2}, \quad (\|\overset{a}{\nabla}u\|_0^2 + \|\overset{v}{\nabla}u\|_0^2 + \|u\|_0^2)^{1/2} \qquad (6.5.23)$$

are equivalent on the subspace, of $H^1_{\mathrm{loc}}(T^0M)$, consisting of functions possessing homogeneity (3.5.6). By (6.5.21) and (6.5.22), the last of these norms is equivalent to $\|\overset{a}{\nabla}u\|_0$. Therefore (6.5.17) implies the estimate

$$\|u\|^2_{H^1(\Omega M)} \le C\left(m\|u\|_0^2 + m\|u\|_0 \cdot \|\delta f\|_0 + N^2(f)\right)$$

that coincides with (6.4.4). The lemma is proved.

Proof of Theorem 6.3.2. In the case of $m = 0$, the theorem was proved at the end of Section 6.4. We consider the case of $m = 1$. Given $f \in C^\infty(\tau'_M)$, we define the function u on T^0M by the same equality (3.5.1). In our case the kinetic equation looks as follows:

$$Hu(x, \xi) = f_i(x)\xi^i, \qquad (6.5.24)$$

and $\overset{v}{\nabla}Hu = f$. Therefore the Pestov identity (6.5.7) has the form

$$2\langle \overset{a}{\nabla}u, f\rangle = |\overset{a}{\nabla}u|^2 + \overset{a}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i.$$

After integration over ΩM this gives

$$\|\overset{a}{\nabla}u\|^2_{L_2} - 2(\overset{a}{\nabla}u, f)_{L_2} + n\|Hu\|^2_{L_2} = -\int\limits_{\partial\Omega M} \langle v, \nu\rangle\, d\Sigma^{2n-2}.$$

Estimating the right-hand side integral as above, we get the inequality

$$\|\overset{a}{\nabla}u\|^2_{L_2} - 2(\overset{a}{\nabla}u, f)_{L_2} + n\|Hu\|^2_{L_2} \le C\|If\|^2_1. \qquad (6.5.25)$$

From (6.5.24), we obtain

$$\|Hu\|^2_{L_2} = \int\limits_{\Omega M} f_i(x)f_j(x)\xi^i\xi^j\, d\Sigma(x, \xi) = \int\limits_{M} f_i(x)f_j(x)\left[\int\limits_{\Omega_x M} \xi^i\xi^j\, d\omega_x(\xi)\right] dV^n(x) = \frac{1}{n}\|f\|^2_{L_2}.$$

With the help of the latter equality, (6.5.25) takes the form

$$\|\overset{a}{\nabla}u - f\|^2_{L_2} \le C\|If\|^2_1. \qquad (6.5.26)$$

This estimate holds for every $f \in C^\infty(\tau'_M)$.

Let now $f \in H^1(\tau'_M)$ be such that $If = 0$. Choose a sequence $f_k \in C^\infty(\tau'_M)$ $(k = 1, 2, \ldots)$ converging to f in $H^1(\tau'_M)$ as $k \to \infty$, and write down estimate (6.5.26) for f_k:

$$\|\overset{a}{\nabla}u_k - f_k\|^2_{L_2} \le C\|If_k\|^2_1. \qquad (6.5.27)$$

The right-hand side of this inequality tends to zero as $k \to \infty$ by continuity of the ray transform, and the left-hand side tends to $\|\overset{a}{\nabla}u - f\|^2_{L_2}$ by Corollary 6.4.2 ,with $u = Lf$. Passing to the limit in (6.5.27) as $k \to \infty$, we obtain

$$\overset{a}{\nabla}u = f. \qquad (6.5.28)$$

Applying the operator $\overset{v}{\nabla}$ to equality (6.5.28) and using permutability of $\overset{v}{\nabla}$ and $\overset{a}{\nabla}$ (formula (6.1.15)), we obtain $\overset{a}{\nabla}_i\overset{v}{\nabla}u = 0$. Contracting this equality with ξ^i, we have $\xi^i\overset{a}{\nabla}_i\overset{v}{\nabla}u = 0$. By (6.1.11), the latter relation can be rewritten in the form

$$(H\overset{v}{\nabla}u)_i - a_i^p\overset{v}{\nabla}_p u = 0.$$

Together with the homogeneous boundary condition $\overset{v}{\nabla}u|_{\partial_-\Omega M} = 0$, the latter equation implies that $\overset{v}{\nabla}u = 0$, i.e., the function u is independent of ξ, $u = u(x)$. Now the kinetic equation (6.5.24) gives us: $f = du$. We have thus proved Theorem 6.3.2 in the case of $k = 1$. As we have seen, this implies the theorem for all $k \ge 1$.

Lecture 7
The inverse problem of determining a source in the transport equation

As we have seen in Section 3.5, the integral geometry problem for a scalar function $f(x)$ is equivalent to the inverse problem of determining a source in the stationary kinetic equation

$$Hu(x, \xi) = f(x).$$

The latter equation has a simple physical meaning: it describes the distribution of particles (or a radiation) moving along geodesics of a given Riemannian metric with unit speed and not interacting with each other and with a medium. If we wish to take account of interaction of particles with the medium, then we have to insert extra summands into the equation. The simplest of such summands describes attenuation of particles by the medium. From the standpoint of integral geometry, the problem consists in inverting the operator that differs from the ray transform (3.3.2) by the presence of the factor $\exp\left[-\int_t^0 \alpha(\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s))\, ds\right]$ in the integrand. The function $\alpha(x, \xi)$ is called the attenuation, and the corresponding integral geometry operator is called the attenuated ray transform. We will denote this operator by I^α. It is the main mathematical subject of emission tomography. Statements of problems of emission tomography can vary considerably. For instance, the problem of simultaneously determining the source f and the attenuation α is of great practical import. We will here deal with a more modest problem of determining the source f on condition that the attenuation α is known. Moreover, we will assume the attenuation $\alpha(x, \xi)$ to be isotropic, i.e., independent of the second argument. We will restrict ourselves to considering the attenuated ray transform of scalar functions, i.e., the case of $m = 0$ in (3.3.2). In the case of $m > 0$ investigation of the attenuated ray transform comes across the next fundamental question: does there exist, for I^α, an analog of the operator d of inner differentiation?

The summand second in complexity which is usually included into the kinetic equation is the scattering integral describing the effects of collision of particles with motionless atoms of the medium. The kinetic equation with the scattering integral is conventionally called the linear transport equation. The latter now has no simple interpretation in terms of integral geometry. Nevertheless, the methods of integral geometry can successfully be used in studying inverse problems for this equation. Below we consider the inverse problem of determining a source in the linear transport equation and prove uniqueness of a solution and a stability estimate for it under some assumptions.

7.1 The transport equation

Recall that, for a Riemannian manifold (M, g), the differential operator $H : C^\infty(\Omega M) \to C^\infty(\Omega M)$ of differentiation along the geodesic flow is defined on the manifold ΩM of unit tangent vectors.

Denote $\Omega^2 M = \{(x; \xi, \xi') \mid x \in M;\ \xi, \xi' \in T_x M,\ |\xi| = |\xi'| = 1\}$, and fix functions $\alpha \in C^\infty(\Omega M)$ and $s \in C^\infty(\Omega^2 M)$, called henceforth the *attenuation* and *scattering diagram* respectively. The equation

$$\left(\frac{\partial}{\partial t} + H + \alpha\right) u(x, \xi, t) = \frac{1}{\omega_{n-1}} \int\limits_{\Omega_x M} s(x, \xi, \xi') u(x, \xi', t)\, d\omega_x(\xi') + f(x, \xi, t) \qquad (7.1.1)$$

on the manifold $\Omega M \times \mathbf{R}$ is referred to as the (*unit-velocity*) *transport equation*. Here $u(x, \xi, t)$ is a sought function; $f(x, \xi, t)$ is a given function, called the *source*; $\Omega_x M = \Omega M \cap T_x M$; $d\omega_x$ is the volume form on the sphere $\Omega_x M$ induced by the metric g; and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the volume of the unit sphere

in $\mathbf{R}^n$. In particular, if the source f and the solution u are independent of time, we have the *stationary* transport equation. To find a solution to equation (7.1.1), we have to specify the initial data $u(x, \xi, 0)$ and the incoming flow $u|_{\partial_- \Omega M \times \mathbf{R}}$. We restrict ourselves to considering the homogeneous initial and boundary data:

$$u(x, \xi, 0) = 0, \tag{7.1.2}$$

$$u|_{\partial_- \Omega M \times \mathbf{R}} = 0. \tag{7.1.3}$$

We will consider the inverse problem for the stationary transport equation, assuming the source and medium isotropic. This means that the source and attenuation are independent of the second argument: $f = f(x)$ and $\alpha = \alpha(x)$, and that the scattering diagram $s(x, \xi, \xi')$ depends only on the angle between the vectors ξ and ξ'; i.e.,

$$s(x, \xi, \xi') = \sigma(x; \langle \xi, \xi' \rangle). \tag{7.1.4}$$

We thus consider the boundary value problem

$$(H + \alpha(x)) u(x, \xi) = \frac{1}{\omega_{n-1}} \int_{\Omega_x M} \sigma(x; \langle \xi, \xi' \rangle) u(x, \xi') \, d\omega_x(\xi') + f(x), \tag{7.1.5}$$

$$u|_{\partial_- \Omega M} = 0 \tag{7.1.6}$$

on the manifold ΩM. We use the outgoing flow

$$u|_{\partial_+ \Omega M} = u_0(x, \xi) \tag{7.1.7}$$

as data for the inverse problem that is formulated as follows: find the function f from the known trace (7.1.7) of a solution to boundary value problem (7.1.5)–(7.1.6).

While dealing with the inverse problem in the current lecture, we will not discuss in detail the questions related to existence of a solution to the direct problem. However, it is impossible to avoid such a discussion completely, since we have to use some properties of a solution to boundary value problem (7.1.5)–(7.1.6) in solving the inverse problem. For that reason, we now discuss this question briefly on an informal level by using physical terms.

It is intuitively clear that, for a solution to boundary problem (7.1.5)–(7.1.6) to exist, the solution of the corresponding nonstationary problem (7.1.1)–(7.1.3) must stabilize as $t \to \infty$. However, there are at least three reasons that may lead to an unbounded growth of energy inside ΩM and, as such, create obstacles to the stabilization of the solution.

The first reason relates to existence of geodesics of infinite length. In such a case some particles do not leave ΩM and can disappear only due to attenuation.

The second reason relates to the fact that the scattering integral on the right-hand side of equation (7.1.1) describes not only the changes in the direction of movement of the particles but also the breeding of particles in collisions with the atoms of the medium. A chain reaction is possible if the scattering diagram is large as compared with the attenuation. In such a case the solution to equation (7.1.1) increases exponentially with time.

Finally, the third reason relates to the possibility of a chain reaction because of geometry of geodesics, i.e., because of focusing geodesics in a small volume.

We exclude the first of the reasons by assuming the manifold (M, g) to be dissipative. The other two reasons are excluded by some assumption of smallness of the scattering diagram in comparison with the attenuation. In the case of an arbitrary metric g, it is a rather difficult problem to find minimal constraints on the attenuation and the scattering diagram which would guarantee existence of a solution to problem (7.1.5)–(7.1.6).

In the case of smooth functions f, α, and σ a solution $u(x, \xi)$ to boundary value problem (7.1.5)–(7.1.6), if exists, is a smooth function on $\Omega M \setminus \partial_0 \Omega M$. Any point of the set $\partial_0 \Omega M$ may be singular for the function $u(x, \xi)$ since some partial derivatives of $u(x, \xi)$ can be unbounded in a neighborhood about the point. Nevertheless, one can show that the singularities are such that all integrals below converge. For the integral geometry problem, such questions were in detail discussed in Section 4.3. For a solution to problem (7.1.5)–(7.1.6), the question is not much harder. Therefore, to simplify presentation, in what follows we pay no attention to the singularities of the function $u(x, \xi)$ and treat the function as belonging to $C^\infty(\Omega M)$.

If $\sigma \equiv 0$ then the boundary value problem (7.1.5)–(7.1.6) has the explicit solution given by the formula

$$u_0(x, \xi) = I^\alpha f(x, \xi) \equiv \int_{\tau_-(x, \xi)}^0 f(\gamma_{x,\xi}(t)) \exp \left[- \int_t^0 \alpha(\gamma_{x,\xi}(s)) \, ds \right] dt \quad ((x, \xi) \in \partial_+ \Omega M), \tag{7.1.8}$$

where $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \xi$. The operator

$$I^\alpha : C^\infty(M) \to C^\infty(\partial_+\Omega M) \tag{7.1.9}$$

defined by (7.1.8) is called the *attenuated ray transform corresponding to the attenuation* α. The operator is easily shown to be extendible to a bounded operator

$$I^\alpha : H^k(M) \to H^k(\partial_+\Omega M)$$

for every $k \geq 0$. The latter plays a key role in problems of emission tomography.

7.2   Statement of the results

Given functions $\alpha \in C^\infty(M)$ and $\sigma \in C^\infty(M \times [-1,1])$, we define the function $\kappa = \kappa[\alpha, \sigma] \in C(M)$ as follows. For $n = \dim M \geq 3$, we expand $\sigma(x; \mu)$ in a Fourier series in Gegenbauer's polynomials:

$$\sigma(x; \mu) = \sum_{k=0}^{\infty} \sigma_k(x) C_k^{(n/2-1)}(\mu), \tag{7.2.1}$$

and put

$$\kappa(x) = \max_{k \geq 1} \left| \frac{n-2}{n+2k-2} \sigma_k(x) - \alpha(x) \right|. \tag{7.2.2}$$

For $n = 2$, formulas (7.2.1) and (7.2.2) are replaced with the next:

$$\sigma(x; \cos\theta) = \sum_{k=-\infty}^{\infty} \sigma_k(x) e^{ik\theta},$$

$$\kappa(x) = \max_{|k| \geq 1} |\sigma_k(x) - \alpha(x)|. \tag{7.2.3}$$

Note that $\kappa(x)$ is independent of $\sigma_0(x)$. In particular, $\kappa(x) = |\alpha(x)|$ if the scattering diagram $\sigma(x; \mu) = \sigma(x)$ does not depend on μ (sometimes such scattering diagrams are called isotropic).

We can now formulate the main assertion of the current lecture.

Theorem 7.2.1 *Let* (M, g) *be a compact dissipative Riemannian manifold of dimension* $n \geq 2$ *and let* $\alpha \in C^\infty(M)$ *and* $\sigma \in C^\infty(M \times [-1,1])$ *be two functions. Assume that, for every* $(x, \xi) \in \Omega M$, *the equation*

$$\frac{D^2\eta}{dt^2} + \overset{\kappa}{R}(t)\eta = 0 \tag{7.2.4}$$

lacks conjugate points on the geodesic $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), \tau_+(x,\xi)] \to M$. *Here* $D/dt = \dot{\gamma}^i \nabla_i$ *is the covariant derivative along* γ, *and* $\overset{\kappa}{R}(t) : T_{\gamma(t)}M \to T_{\gamma(t)}M$ *is the linear operator whose matrix is defined in local coordinates by the equality*

$$\overset{\kappa}{R}{}_k^p(t) = \left[g^{pi} \left(R_{ijkl} + \kappa^2 (g_{ik}g_{jl} - g_{il}g_{jk}) \right) \right]_{x=\gamma(t)} \dot{\gamma}^j(t) \dot{\gamma}^l(t), \tag{7.2.5}$$

where (R_{ijkl}) *is the curvature tensor and the function* $\kappa(x)$ *is defined by* (7.2.1)–(7.2.3). *Then every function* $f \in H^1(M)$ *can be uniquely recovered from trace* (7.1.7) *of a solution to boundary value problem* (7.1.5)–(7.1.6), *and the stability estimate*

$$\|f\|_{L_2(M)} \leq C\|u_0\|_{H^1(\partial_+\Omega M)} \tag{7.2.6}$$

holds with some constant C *independent of* f.

We now formulate some corollaries of the theorem which are related to the cases in which either the scattering integral is absent or the metric g is Euclidean. Both cases are significant for applications.

Corollary 7.2.2 *Let* (M, g) *be a CDRM and* $\alpha \in C^\infty(M)$. *Assume that equation* (7.2.4) *with*

$$\overset{\kappa}{R}{}_k^p(t) = \overset{\alpha}{R}{}_k^p(t) = [g^{pi}(R_{ijkl} + |\alpha|^2(g_{ik}g_{jl} - g_{il}g_{jk}))]_{x=\gamma(t)} \dot{\gamma}^j(t) \dot{\gamma}^l(t)$$

lacks conjugate points on the geodesic $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), \tau_+(x,\xi)] \to M$ *for every* $(x,\xi) \in \Omega M$. *Then the operator*

$$I^\alpha : H^1(M) \to H^1(\partial_+\Omega M) \tag{7.2.7}$$

is injective and the stability estimate

$$\|f\|_{L_2(M)} \le C\|I^\alpha f\|_{H^1(\partial_+\Omega M)} \tag{7.2.8}$$

holds with some constant C *independent of* f.

In the case of $\alpha \equiv \sigma \equiv 0$ equation (7.2.4) transforms into the classical Jacobi equation

$$\frac{D^2\eta}{dt^2} + R(\dot\gamma, \eta)\dot\gamma = 0, \tag{7.2.9}$$

and operator (7.2.7) coincides with the ray transform (3.3.3) for $m = 0$. In this case Corollary 7.2.2 coincides with the claim of Theorem 6.3.2 for $m = 0$.

We now discuss in brief the role of the curvature tensor in Theorem 7.2.1 and Corollary 7.2.2. It is well known [33] that, if all sectional curvatures are nonpositive, then the Jacobi equation lacks conjugate points on a geodesic segment of any length. Of course, this property may fail when we add the summand with the factor κ^2 to the right-hand side of (7.2.5). Nevertheless, the general tendency remains preserved: the more negative the sectional curvature is, the larger values κ^2 may assume without violating the assumptions of Theorem 7.2.1. Thus, there appears an original phenomenon when the negative values of the curvature compensate attenuation and scattering.

We now consider the case in which M is a bounded domain in $\mathbf{R}^n$, and the metric g coincides with the Euclidean metric. In this case equation (7.1.5) becomes the classical transport equation

$$\xi^i \frac{\partial u(x,\xi)}{\partial x^i} + \alpha(x)u(x,\xi) = \frac{1}{\omega_{n-1}} \int\limits_{|\xi'|=1} \sigma(x; \langle \xi, \xi' \rangle)u(x,\xi')\,d\xi' + f(x), \tag{7.2.10}$$

and system (7.2.4) is reduced to the single scalar equation

$$\frac{d^2\eta}{dt^2} + \kappa^2\eta = 0. \tag{7.2.11}$$

We thus obtain

Corollary 7.2.3 *Let* M *be a closed bounded domain in* $\mathbf{R}^n$ *with smooth strictly convex boundary. Let functions* $\alpha \in C^\infty(M)$ *and* $\sigma \in C^\infty(M \times [-1,1])$ *be such that equation* (7.2.11) *lacks conjugate points on any straight line segment* $\gamma : [a,b] \to M$; *here* $\kappa = \kappa[\alpha,\sigma]$ *is defined by formulas* (7.2.1) *and* (7.2.2). *Then every function* $f \in H^1(M)$ *is uniquely recovered from trace* (7.1.7) *of the solution to boundary value problem* (7.2.10), (7.1.6) *and stability estimate* (7.2.6) *is valid.*

Finally, if $\sigma \equiv 0$ then the boundary value problem (7.2.10), (7.1.6) is explicitly solvable; and we thus obtain the assertion of invertibility of the attenuated ray transform on Euclidean space. In this case the transform is conveniently written down as

$$I^\alpha f(x,\xi) = \int\limits_{-\infty}^{\infty} f(x+t\xi) \exp\left[-\int\limits_{t}^{\infty} \alpha(x+s\xi)\,ds \right] dt \quad (x \in \mathbf{R}^n,\ 0 \ne \xi \in \mathbf{R}^n), \tag{7.2.12}$$

on assuming that the functions f and α are extended by zero outside M. Equation (7.2.11) takes the form

$$\frac{d^2\eta}{dt^2} + |\alpha|^2\eta = 0. \tag{7.2.13}$$

A number of conditions are known which ensure the absence of conjugate points for a scalar equation. Some of them are based on the Sturm comparison theorems, and the others, on Lyapunov's integral estimates [37]. The simplest of them guarantees the absence of conjugate points for equation (7.2.11) if the inequality

$$\kappa_0 \operatorname{diam} M < \pi$$

is valid with $\kappa_0 = \sup\limits_{x \in M} \kappa(x)$, $\operatorname{diam} M = \sup\limits_{x,y \in M} |x - y|$. In particular, for equation (7.2.13), this condition takes the form

$$\alpha_0 \operatorname{diam} M < \pi, \tag{7.2.14}$$

where $\alpha_0 = \sup\limits_{x \in M} |\alpha(x)|$.

7.3 Proof of Theorem 7.2.1

Equation (7.1.5) is originally considered on ΩM. For convenience (to have the possibility of applying the partial derivatives $\partial/\partial\xi^i$) we extend the equation to $T^0 M$ in such a way that all its terms become positively homogeneous functions of zero degree in ξ. Since H increases the degree of homogeneity in ξ by one, we extend the function $u(x, \xi)$ to $T^0 M$ by putting

$$u(x, \lambda\xi) = \lambda^{-1} u(x, \xi) \quad (\lambda > 0). \tag{7.3.1}$$

Introducing the notation

$$Su(x, \xi) = \frac{1}{\omega_{n-1}} \int\limits_{\Omega_x M} \sigma\left(x; \langle \xi/|\xi|, \xi' \rangle\right) u(x, \xi') \, d\omega_x(\xi') \tag{7.3.2}$$

for the scattering integral and inserting the factor $|\xi|$ into the second summand on the left-hand side of (7.1.5), we obtain the equation

$$Hu(x, \xi) + \alpha(x)|\xi|u(x, \xi) = Su(x, \xi) + f(x) \tag{7.3.3}$$

which holds on $T^0 M$.

Let $a \in C^\infty(\beta_0^2 M; T^0 M)$ be some semibasic tensor field on $T^0 M$ satisfying (6.1.4)–(6.1.6); we shall specify the choice of the field later. Let $\overset{a}{\nabla}$ be the corresponding modified horizontal derivative. We introduce semibasic vector fields y and z on $T^0 M$ by the equalities

$$\overset{v}{\nabla}u = -\frac{u}{|\xi|^2}\xi + y, \tag{7.3.4}$$

$$\overset{a}{\nabla}u = \frac{Hu}{|\xi|^2}\xi + z. \tag{7.3.5}$$

Observe that

$$\langle y, \xi \rangle = \langle z, \xi \rangle = 0. \tag{7.3.6}$$

To verify (7.3.6), it suffices to take the scalar products of equalities (7.3.4) and (7.3.5) with ξ and use relations (6.1.12) and (7.3.1). In particular, (7.3.5) implies

$$|\overset{a}{\nabla}u|^2 = \frac{1}{|\xi|^2}|Hu|^2 + |z|^2. \tag{7.3.7}$$

We apply the operator $\overset{v}{\nabla}$ to equation (7.3.3):

$$\overset{v}{\nabla}(Hu) + \alpha|\xi|\overset{v}{\nabla}u + \frac{\alpha u}{|\xi|}\xi = \overset{v}{\nabla}(Su).$$

Inserting expression (7.3.4) for $\overset{v}{\nabla}u$ into the preceding equality, we obtain

$$\overset{v}{\nabla}(Hu) = -\alpha|\xi|y + \overset{v}{\nabla}(Su). \tag{7.3.8}$$

By (7.3.6), the first summand on the right-hand side of (7.3.8) is orthogonal to the vector ξ. The same is true for the second summand, since the function Su is homogeneous of zero degree. Taking the scalar product of (7.3.8) and $\overset{a}{\nabla}u = z + (Hu)\xi/|\xi|^2$, we obtain

$$\langle \overset{a}{\nabla}u, \overset{v}{\nabla}(Hu) \rangle = \langle z, \overset{v}{\nabla}(Su) - \alpha|\xi|y \rangle. \tag{7.3.9}$$

For the function u, Pestov's identity (6.1.28) holds on $T^0 M$, with the semibasic fields (v_i) and (w^i) defined by formulas (6.1.29) and (6.1.30). Comparing (6.1.28) and (7.3.9), we find

$$|\overset{a}{\nabla} u|^2 = 2\langle z, \overset{v}{\nabla}(Su) - \alpha|\xi|y\rangle + \overset{a}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u - \overset{a}{\nabla}^i v_i - \overset{v}{\nabla}_i w^i. \tag{7.3.10}$$

We transform the second summand on the right-hand side of (7.3.10) by using properties (6.1.24) and (6.1.25) of the curvature tensor to obtain

$$\overset{a}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}^j u\overset{v}{\nabla}^j u = \overset{a}{R}_{ijkl}\xi^i\xi^k\left(y^j - \frac{u}{|\xi|^2}\xi^j\right)\left(y^l - \frac{u}{|\xi|^2}\xi^l\right) = \overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l.$$

Owing to the last relation and (7.3.7), formula (7.3.10) takes the form

$$\frac{1}{|\xi|^2}|Hu|^2 + |z|^2 = 2\langle z, \overset{v}{\nabla}(Su) - \alpha|\xi|y\rangle + \overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l - \overset{a}{\nabla}^i v_i - \overset{v}{\nabla}_i w^i.$$

We multiply this equality by the volume form $d\Sigma = d\Sigma^{2n-1}$ of the manifold ΩM, integrate the result over ΩM, and transform the integrals of the terms of divergence type by the Gauss-Ostrogradskiĭ formula (6.1.32) for the modified horizontal derivative and formula (2.7.1) for the vertical derivative. As a result, we obtain

$$\int\limits_{\Omega M} (|Hu|^2 + |z|^2)\,d\Sigma = 2\int\limits_{\Omega M} \langle z, \overset{v}{\nabla}(Su) - \alpha y\rangle\,d\Sigma +$$

$$+ \int\limits_{\Omega M} \overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l\,d\Sigma - \int\limits_{\partial\Omega M} \langle v, \nu\rangle\,d\Sigma^{2n-2} - (n-2)\int\limits_{\Omega M} \langle w, \xi\rangle\,d\Sigma. \tag{7.3.11}$$

The coefficient of the last summand is written down on account of the homogeneity of w which insures from (6.1.30) and (7.3.1). Furthermore, (6.1.30) implies that $\langle w, \xi\rangle = |Hu|^2$. By (2.7.29), the volume form $d\Sigma$ can be represented as $d\Sigma(x,\xi) = d\omega_x(\xi) \wedge dV^n(x)$, where dV^n is the Riemannian volume form on M. Thus, equality (7.3.11) takes the form

$$\int\limits_{\Omega M} [(n-1)|Hu|^2 + |z|^2]\,d\Sigma = 2\int\limits_{M}\left[\int\limits_{\Omega_x M} \langle z, \overset{v}{\nabla}(Su) - \alpha y\rangle\,d\omega_x(\xi)\right] dV^n(x) +$$

$$+ \int\limits_{\Omega M} \overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l\,d\Sigma - \int\limits_{\partial\Omega M} \langle v, \nu\rangle\,d\Sigma^{2n-2}. \tag{7.3.12}$$

We use the following claim:

Lemma 7.3.1 *The inner integral of the first summand on the right-hand side of (7.3.12) can be estimated as follows:*

$$2\left|\int\limits_{\Omega_x M} \langle z, \overset{v}{\nabla}(Su) - \alpha y\rangle\,d\omega_x(\xi)\right| \leq \int\limits_{\Omega_x M} (|z(x,\xi)|^2 + \kappa^2(x)|y(x,\xi)|^2)\,d\omega_x(\xi), \tag{7.3.13}$$

where the function $\kappa = \kappa[\alpha, \sigma]$ is defined by formulas (7.2.1) and (7.2.2).

We postpone the proof of the lemma to the end of the section. Now we continue proving Theorem 7.2.1 with the help of the lemma.

Estimating the first summand on the right-hand side of (7.3.12) with the help of (7.3.13), we obtain the inequality

$$(n-1)\int\limits_{\Omega M} |Hu|^2\,d\Sigma \leq \int\limits_{\Omega M} (\overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l + \kappa^2|y|^2)\,d\Sigma - \int\limits_{\partial\Omega M} \langle v, \nu\rangle\,d\Sigma^{2n-2}. \tag{7.3.14}$$

We now specify the choice of the tensor field a. To this end, we observe that, in view of the equalities $|\xi| = 1$ and $\langle y, \xi\rangle = 0$, the integrand of the first summand on the right-hand side of (7.3.14) can be represented as

$$\overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l + \kappa^2|y|^2 = (\overset{a}{R}_{ijkl} + \kappa^2(g_{ik}g_{jl} - g_{il}g_{jk}))\xi^i\xi^k y^j y^l. \tag{7.3.15}$$

We wish to choose the field a in such a way that expression (7.3.15) were identically zero. Here we come to a difficulty related to the fact that, by (7.2.1) and (7.2.2), the function $\kappa(x)$ is merely continuous on

M but not differentiable. However, we essentially used the first-order derivatives with respect to x in the construction of the operator $\overset{a}{\nabla}$ and in the proof of Theorem 6.2.1 (for instance, in definition (6.1.20) of the curvature tensor). Therefore, we will proceed as follows: We choose a small number $\delta > 0$ and approximate the function κ by some smooth function $\tilde{\kappa} \in C^\infty(M)$ so as to have

$$|\tilde{\kappa}(x) - \kappa(x)| < \delta. \tag{7.3.16}$$

If we replace κ on the right-hand side of (7.2.5) with $\tilde{\kappa}$, then equation (7.2.4) lacks conjugate points for a sufficiently small δ. Therefore, the conditions of Theorem 6.2.1 are satisfied for the smooth field $S_{ijkl} = \tilde{\kappa}^2(g_{ik}g_{jl} - g_{il}g_{jk})$. Applying Theorem 6.2.1, we find some field $a \in C^\infty(\beta_0^2 M; T^0 M)$ satisfying the relation

$$(\overset{a}{R}_{ijkl} + \tilde{\kappa}^2(g_{ik}g_{jl} - g_{il}g_{jk}))\xi^i\xi^k = 0.$$

Hence, expression (7.3.15) admits the estimate

$$|\overset{a}{R}_{ijkl}\xi^i\xi^k y^j y^l + \kappa^2|y|^2| \le \delta|y|^2 \quad (|\xi| = 1).$$

Using it, (7.3.14) implies the inequality

$$(n-1)\int_{\Omega M} |Hu|^2\,d\Sigma \le \delta\int_{\Omega M} |y|^2\,d\Sigma - \int_{\partial\Omega M} \langle v, \nu\rangle\,d\Sigma^{2n-2}. \tag{7.3.17}$$

By (7.3.4), the field y and the function $Hu = \xi^i\overset{h}{\nabla}_i u$ are independent of the field a involved in the definition of the modified horizontal derivative. The field a participates in (7.3.17) only through the field v determined by formula (6.1.29). The passage to the limit in (7.3.14) as $\delta \to 0$ becomes possible if we convince ourselves that the integrand $\langle v, \nu\rangle$ can be estimated uniformly in δ.

In a neighborhood about an arbitrary point $x_0 \in \partial M$, we can introduce a semigeodesic coordinate system $(x^1, \ldots, x^n)$ in which the boundary is determined by the equation $x^n = 0$, $g_{in} = \delta_{in}$, and the vector ν has coordinates $(0, \ldots, 0, 1)$. Now (6.1.29) implies that

$$\langle v, \nu\rangle = v_n = Lu \equiv \xi_n\overset{a}{\nabla}^\beta u \cdot \overset{v}{\nabla}_\beta u - \xi_\beta\overset{v}{\nabla}_n u \cdot \overset{a}{\nabla}^\beta u. \tag{7.3.18}$$

Moreover, the summation over the index β is taken from 1 to $n-1$. It is essential that the derivative $\overset{a}{\nabla}^n u$ is not involved in Lu. If $(y^1, \ldots, y^{2n-2})$ is a local coordinate system on $\partial\Omega M$, then Lu is a quadratic form in the variables u, $\partial u/\partial y^i$, and $\partial u/\partial|\xi|$. By homogeneity (7.3.1), $\partial u/\partial|\xi| = -u$, and hence L is a first-order quadratic differential operator on the manifold $\partial\Omega M$. Therefore, the following estimate is valid:

$$\left|\int_{\partial\Omega M} \langle v, \nu\rangle\,d\Sigma^{2n-2}\right| \le C\|u|_{\partial\Omega M}\|_{H^1(\partial\Omega M)}^2. \tag{7.3.19}$$

We will demonstrate that the constant C in (7.3.19) can be chosen independently of the number δ involved in (7.3.16); i.e., the coefficients of operator (7.3.18) are bounded uniformly in δ. By the definition of the modified derivative, we have $\overset{a}{\nabla}^i u = \overset{h}{\nabla}^i u + a^{ip}\overset{v}{\nabla}_p u$. To prove our claim, it therefore suffices to estimate the field a uniformly in δ. To this end, we have to return to the proof of Theorem 6.2.1. In the proof we in fact constructed some operator $S \mapsto a$ defined on the set of the fields S satisfying the conditions of the theorem. Tracing the construction of the operator, we can easily see that the operator is continuous in the C-norm.

Thus, the constant C in (7.3.19) is independent of δ. Estimating the last summand on the right-hand side of (7.3.17) with the help of (7.3.19) and passing to the limit as $\delta \to 0$, we obtain

$$\int_{\Omega M} |Hu|^2\,d\Sigma \le C\|u|_{\partial\Omega M}\|_{H^1(\partial\Omega M)}^2.$$

Recalling boundary conditions (7.1.6) and (7.1.7), we get

$$\int_{\Omega M} |Hu|^2\,d\Sigma \le C\|u_0\|_{H^1(\partial_+\Omega M)}^2. \tag{7.3.20}$$

To finish the proof of Theorem 7.2.1, we are left with estimating $\|f\|_{L_2(M)}$ by means of $\int_{\Omega M} |Hu|^2 \, d\Sigma$. To this end, we observe that equation (7.1.5) implies the estimate

$$\|f\|_{L_2(M)}^2 \le \int_{\Omega M} |Hu|^2 \, d\Sigma + C_1 \int_{\Omega M} |u|^2 \, d\Sigma \qquad (7.3.21)$$

with some constant C_1 independent of f. On use made of the Poincaré inequality (see Lemma 4.2.1), boundary condition (7.1.6) implies the estimate

$$\int_{\Omega M} |u|^2 \, d\Sigma \le C_2 \int_{\Omega M} |Hu|^2 \, d\Sigma.$$

The latter together with (7.3.21) gives

$$\|f\|_{L_2(M)}^2 \le C_2 \int_{\Omega M} |Hu|^2 \, d\Sigma.$$

Comparing the last inequality with (7.3.20), we arrive at (7.2.6). The theorem is proved.

Proof of Lemma 7.3.1. We will prove the claim only for $n \ge 3$. In the case of $n = 2$ the proof is similar. To simplify notation, we will not explicitly indicate the point $x \in M$ in the arguments; the point is fixed in the proof.

Our nearest aim is to express $\overset{v}{\nabla}(Su)$ in terms of y. To this end, we rewrite definition (7.3.2) of the scattering integral as

$$Su(\xi) = \frac{1}{\omega_{n-1}} \int_{-1}^{1} (1 - \mu^2)^{(n-3)/2} \sigma(\mu) \, d\mu \int_{\Omega_x^\xi M} u(\mu\xi/|\xi| + \sqrt{1 - \mu^2}\eta) \, d\omega_x^{n-2}(\eta), \qquad (7.3.22)$$

where

$$\Omega_x^\xi M = \{\eta \in T_x M \mid |\eta| = 1, \ \langle \xi, \eta \rangle = 0\}.$$

We represent (7.3.22) as an integral over some domain independent of ξ. To do this, we fix $\xi_0 \in \Omega_x M$ and, for a vector $\xi \in T_x M$ close enough to ξ_0, consider the isometry $\Omega_x^\xi M \to \Omega_x^{\xi_0} M$, $\eta \mapsto \eta'$, defined by the formulas

$$\eta' = \eta - \frac{\langle \xi_0, \eta \rangle}{|\xi| + \langle \xi_0, \xi \rangle}(\xi + |\xi|\xi_0), \quad \eta = \eta' - \frac{\langle \xi, \eta' \rangle}{|\xi|(|\xi| + \langle \xi_0, \xi \rangle)}(\xi + |\xi|\xi_0). \qquad (7.3.23)$$

Changing the integration variable in (7.3.22) in accord with formula (7.3.23), we obtain

$$Su(\xi) = \frac{1}{\omega_{n-1}} \int_{-1}^{1} (1 - \mu^2)^{(n-3)/2} \sigma(\mu) \, d\mu \int_{\Omega_x^{\xi_0} M} u\left(\mu e + \sqrt{1 - \mu^2}\eta' - \sqrt{1 - \mu^2}\frac{\langle e, \eta' \rangle}{1 + \langle \xi_0, e \rangle}(e + \xi_0)\right) d\omega_x^{n-2}(\eta'),$$

where $e = \xi/|\xi|$. Differentiating this equality and putting $\xi_0 = \xi$ in the resulting relation, we obtain

$$\overset{v}{\nabla}_i Su(\xi) = \frac{1}{\omega_{n-1}} \int_{-1}^{1} (1 - \mu^2)^{(n-3)/2} \sigma(\mu) \, d\mu \int_{\Omega_x^\xi M} \left[\mu\delta_i^j - (\mu\xi_i + \sqrt{1 - \mu^2}\eta_i)\xi^j\right] \overset{v}{\nabla}_j u(\mu\xi + \sqrt{1 - \mu^2}\eta) \, d\omega_x^{n-2}(\eta).$$

Returning to the integration variable $\xi' = \mu\xi + \sqrt{1 - \mu^2}\eta$, we write down the obtained result as

$$\overset{v}{\nabla}_i Su(\xi) = \frac{1}{\omega_{n-1}} \int_{\Omega_x M} \sigma(\langle \xi, \xi' \rangle)(\langle \xi, \xi' \rangle \delta_i^j - \xi'_i \xi^j) \overset{v}{\nabla}_j u(\xi') \, d\omega_x(\xi').$$

Inserting expression (7.3.4) for $\overset{v}{\nabla} u$ into the preceding equality, we arrive at the sought representation for $\overset{v}{\nabla}_i Su$ in terms of y:

$$\overset{v}{\nabla}_i Su(\xi) = \frac{1}{\omega_{n-1}} \int_{\Omega_x M} \sigma(\langle \xi, \xi' \rangle)(\langle \xi, \xi' \rangle \delta_i^j - \xi'_i \xi^j) \, y_j(\xi') \, d\omega_x(\xi'). \qquad (7.3.24)$$

We take the scalar product of (7.3.24) and $z(\xi)$ and integrate the result over $\Omega_x M$:

$$\int_{\Omega_x M} \langle z, \overset{v}{\nabla} Su \rangle \, d\omega_x = \frac{1}{\omega_{n-1}} \int_{\Omega_x M} \int_{\Omega_x M} \sigma(\langle \xi, \xi' \rangle) \left(\langle \xi, \xi' \rangle \langle z(\xi), y(\xi') \rangle - \langle z(\xi), \xi' \rangle \langle \xi, y(\xi') \rangle \right) d\omega_x(\xi) \, d\omega_x(\xi').$$

The last relation can be rewritten in the more convenient form

$$\int_{\Omega_x M} \langle z, \overset{v}{\nabla} Su \rangle \, d\omega_x = \frac{1}{\omega_{n-1}} \int_{\Omega_x M} \int_{\Omega_x M} \sigma(\langle \xi, \xi' \rangle) \langle \xi \wedge z(\xi), \xi' \wedge y(\xi') \rangle \, d\omega_x(\xi) \, d\omega_x(\xi'). \tag{7.3.25}$$

Observe that the mean of the bivector $\xi \wedge y(\xi)$ on the sphere $\Omega_x M$ is equal to zero; i.e.,

$$\frac{1}{\omega_{n-1}} \int_{\Omega_x M} \xi \wedge y(\xi) \, d\omega_x(\xi) = 0. \tag{7.3.26}$$

Indeed, by (7.3.4), $\xi \wedge y(\xi) = \xi \wedge \overset{v}{\nabla} u(\xi)$. Therefore, equality (7.3.26) amounts to the following:

$$\frac{1}{\omega_{n-1}} \int_{\Omega_x M} \xi \wedge \overset{v}{\nabla} u(\xi) \, d\omega_x(\xi) = 0,$$

which is easily seen to be valid for every function $u(\xi)$.

We choose an orthonormal basis $\eta_1, \dots, \eta_N$ for the space $\Lambda^2 T_x M$ of bivectors and expand $\xi \wedge z(\xi)$ and $\xi \wedge y(\xi)$ in the basis:

$$\xi \wedge z(\xi) = \sum_{\beta=1}^{N} z^\beta(\xi) \eta_\beta, \quad \xi \wedge y(\xi) = \sum_{\beta=1}^{N} y^\beta(\xi) \eta_\beta.$$

Now formula (7.3.25) takes the form

$$\int_{\Omega_x M} \langle z, \overset{v}{\nabla} Su \rangle \, d\omega_x = \sum_{\beta=1}^{N} \frac{1}{\omega_{n-1}} \int_{\Omega_x M} \int_{\Omega_x M} \sigma(\langle \xi, \xi' \rangle) z^\beta(\xi) y^\beta(\xi') \, d\omega_x(\xi) \, d\omega_x(\xi'). \tag{7.3.27}$$

Since ξ is orthogonal to $z(\xi)$ and $y(\xi)$, the equality $\langle z(\xi), y(\xi) \rangle = \langle \xi \wedge z(\xi), \xi \wedge y(\xi) \rangle$ holds and, consequently,

$$\int_{\Omega_x M} \langle z, y \rangle \, d\omega_x = \sum_{\beta=1}^{N} \int_{\Omega_x M} z^\beta(\xi) y^\beta(\xi) \, d\omega_x(\xi). \tag{7.3.28}$$

We expand each of the functions $z^\beta(\xi)$ and $y^\beta(\xi)$ in Fourier series in spherical harmonics:

$$z^\beta(\xi) = \sum_{k=0}^{\infty} z_k^\beta(\xi), \quad y^\beta(\xi) = \sum_{k=1}^{\infty} y_k^\beta(\xi). \tag{7.3.29}$$

Pay attention to the fact that the second of expansions (7.3.29) starts with $k = 1$ because of (7.3.26). Applying the multidimensional version of the Funk-Hecke theorem (Theorem XI.4 of [83]), we express integrals (7.3.27) and (7.3.28) in terms of expansions (7.2.1) and (7.3.29):

$$\int_{\Omega_x M} \langle z, \overset{v}{\nabla} Su \rangle \, d\omega_x = \sum_{\beta=1}^{N} \sum_{k=1}^{\infty} \frac{n-2}{n+2k-2} \sigma_k \int_{\Omega_x M} z_k^\beta(\xi) y_k^\beta(\xi) \, d\omega_x(\xi),$$

$$\int_{\Omega_x M} \langle z, y \rangle \, d\omega_x = \sum_{\beta=1}^{N} \sum_{k=1}^{\infty} \int_{\Omega_x M} z_k^\beta(\xi) y_k^\beta(\xi) \, d\omega_x(\xi).$$

Hence,

$$\int_{\Omega_x M} \langle z, \overset{v}{\nabla} Su - \alpha y \rangle \, d\omega_x = \sum_{\beta=1}^{N} \sum_{k=1}^{\infty} \left(\frac{n-2}{n+2k-2} \sigma_k - \alpha \right) \int_{\Omega_x M} z_k^\beta(\xi) y_k^\beta(\xi) \, d\omega_x(\xi).$$

This implies the inequality

$$2\left|\int\limits_{\Omega_x M} \langle z, \overset{v}{\nabla} Su - \alpha y\rangle \, d\omega_x\right| \leq \sum_{\beta=1}^{N} \sum_{k=1}^{\infty} \int\limits_{\Omega_x M} \left(\left|z_k^\beta\right|^2 + \left|\frac{n-2}{n+2k-2}\sigma_k - \alpha\right|^2 \left|y_k^\beta\right|^2\right) d\omega_x(\xi).$$

Defining the number κ by formula (7.2.2), we obtain

$$2\left|\int\limits_{\Omega_x M} \langle z, \overset{v}{\nabla} Su - \alpha y\rangle \, d\omega_x\right| \leq \sum_{\beta=1}^{N} \sum_{k=1}^{\infty} \int\limits_{\Omega_x M} \left(\left|z_k^\beta\right|^2 + \kappa^2\left|y_k^\beta\right|^2\right) d\omega_x(\xi) = \int\limits_{\Omega_x M} \left(|z|^2 + \kappa^2|y|^2\right) d\omega_x(\xi).$$

The lemma is proved.

7.4   Some remarks

The problem of emission tomography is thoroughly investigated in the case when the metric is Euclidean and the attenuation α is constant [63, 76, 85, 3]. In the case of the Euclidean metric and nonconstant attenuation (Corollary 7.2.3 relates to this case), as far as the author knows, all results are obtained under some assumptions on smallness of the attenuation α or the domain M [67, 49, 39]. Of particular interest is the paper [26] by D. Finch, where uniqueness is proved under assumption (7.2.14) in which the right-hand side is replaced with 5.37.

In [78] the author obtained some result that is rougher than Theorem 7.2.1 but applicable in the more general situation when the scattering diagram $s(x, \xi, \xi')$ depends on all variables.

Transport equation (7.1.5) describes a distribution of particles moving along geodesics of a Riemannian metric with unit velocity. This fact can be expressed in physical terms as follows: the particle movement is determined by the Hamiltonian $\mathcal{H}(q, p) = \frac{1}{2}g^{ij}(q)p_i p_j$ quadratic in the impulse p. The question arises of extending the methods and results to the case of more general Hamiltonians. In [79] such a generalization is obtained for Hamiltonians that are convex and positively homogeneous in the impulse. From the geometrical viewpoint, this means considering a Finsler metric instead of Riemannian one. It turns out that almost all our techniques can be generalized to the Finsler case.

Lecture 8
Integral geometry on Anosov manifolds

Till now we investigated integral geometry on manifolds with boundary. Here we are interesting in periodical problems like the following one. Let (M, g) be a closed (= compact without boundary) Riemannian manifold. To which extent is a smooth function $f \in C^\infty(M)$ determined by its integrals over all closed geodesics?

Of course, the question is sensible only in the case when (M, g) has sufficiently many closed geodesics. Therefore the question was first investigated for symmetric spaces of rank one. The simplest of such manifolds is the sphere with the standard metric. P. Funk [28] proved that the even part of a function on the two-dimensional sphere is determined by integrals over great circles. This work is traditionally considered as the start point of integral geometry. Later the problem was investigated for some other symmetric spaces of rank one.

Closed Riemannian manifolds of negative curvature constitute another natural class of manifolds with sufficiently rich family of closed geodesics. For such a manifold, the set of closed geodesics is dense in the set of all geodesics. Integral geometry on negatively curved manifolds is of great interest because, first of all, it closely relates to the classical problem of spectral rigidity.

As is known, the geodesic flow of a negatively curved manifold is of Anosov type. Therefore closed Riemannian manifolds with geodesic flow of Anosov type (we call them Anosov manifolds) constitute the natural generalization of the class of negatively curved manifold. For Anosov manifolds, we prove periodical analogs of theorems 3.4.3, 4.4.1, and 6.3.1.

In this lecture we follow the papers [21], [22], and [81].

8.1 Posing the problem and formulating results

Let (M, g) be a closed Riemannian manifold. For a symmetric tensor field $f \in C^\infty(S^m \tau'_M)$ and a closed geodesic $\gamma : [a, b] \to M$, we may consider the integral

$$If(\gamma) = \oint_\gamma \langle f, \dot\gamma^m \rangle dt = \int_a^b f_{i_1 \ldots i_m}(\gamma(t)) \dot\gamma^{i_1}(t) \ldots \dot\gamma^{i_m}(t) \, dt. \tag{8.1.1}$$

The integrand on (8.1.1) is written with use made by local coordinates. Nevertheless, it is evidently invariant, i.e., independent of the choice of coordinates. Let $Z^\infty(S^m \tau'_M)$ denote the subspace of $C^\infty(S^m \tau'_M)$ consisting of all fields f such that $If(\gamma) = 0$ for every closed geodesic γ. For $m > 0$ this subspace is not zero as is seen from the following argument. A tensor field f is called the *potential field* if it can be represented in the form $f = dv$ for some $v \in C^\infty(S^{m-1} \tau'_M)$. Here d is the inner differentiation defined in Section 2.4. Let $P^\infty(S^m \tau'_M)$ denote the space of all potential fields. If $f = dv$, then the integrand on (8.1.1) equals to $d(v_{i_1 \ldots i_{m-1}}(\gamma(t)) \dot\gamma^{i_1}(t) \ldots \dot\gamma^{i_{m-1}}(t))/dt$. Therefore there is the inclusion

$$P^\infty(S^m \tau'_M) \subset Z^\infty(S^m \tau'_M). \tag{8.1.2}$$

The principal question of integral geometry of tensor fields is formulated as follows: for what classes of closed Riemannian manifolds and for what values of m is inclusion (8.1.2) in fact the equality?

We remind the definition of an Anosov flow. Let $H \in C^\infty(\tau_N)$ be a vector field, on a closed manifold N, not vanishing at any point, and $G^t : N \to N$ be the one-parameter group of diffeomorphisms (or flow)

generated by the vector field. G^t is called the *Anosov flow* if, for every point $x \in N$, the tangent space $T_x N$ splits into the direct sum of three subspaces

$$T_x N = \{H(x)\} \oplus X^s(x) \oplus X^u(x),$$

where $\{H(x)\}$ is the one-dimensional subspace spanned by the vector $H(x)$, and two other subspaces are such that for $\xi \in X^s(x)$, $\eta \in X^u(x)$ the differential $d_x G^t$ satisfies the estimates

$$|(d_x G^t)\xi| \leq ae^{-ct}|\xi| \quad \text{for} \quad t > 0, \qquad |(d_x G^t)\xi| \geq be^{-ct}|\xi| \quad \text{for} \quad t < 0;$$

$$|(d_x G^t)\eta| \leq ae^{ct}|\eta| \quad \text{for} \quad t < 0, \qquad |(d_x G^t)\eta| \geq be^{ct}|\eta| \quad \text{for} \quad t > 0,$$

where a, b, c are positive constants independent of x, ξ, η.

If such a splitting exists, then it is unique, and $\dim X^s(x)$ is independent of x. The subspaces X^s and X^u are called the *stable* and *unstable subspaces* respectively.

An *Anosov manifold* is a closed Riemannian manifold whose geodesic flow $G^t : \Omega M \to \Omega M$ is of Anosov type. The following two claims are valid for such a manifold: (1) the orbit of a point (x, ξ) with respect to the geodesic flow is dense in ΩM for almost all $(x, \xi) \in \Omega M$; (2) the set of $(x, \xi) \in \Omega M$, such that the geodesic $\gamma_{x,\xi}$ is closed, is dense in ΩM. See [11] for proofs. A closed Riemannian manifold of negative sectional curvature is an Anosov manifold, and the class of Anosov manifolds is wider than the class of closed negatively curved manifolds.

We will return to studying geodesic flows of Anosov type in Section 8.6. Now we formulate main results of the current lecture.

Theorem 8.1.1 *Let (M, g) be an Anosov manifold. If a function $f \in C^\infty(M)$ integrates to zero over every closed geodesic then f must itself be zero.*

The similar result is valid for 1-forms.

Theorem 8.1.2 *Let (M, g) be an Anosov manifold, and f be a smooth 1-form on M. If f integrates to zero around every closed geodesic, then f is an exact form.*

For tensor fields of higher degree we have the weaker result.

Theorem 8.1.3 *For an Anosov manifold of nonpositive sectional curvature, the equality*

$$P^\infty(S^m \tau_M') = Z^\infty(S^m \tau_M')$$

holds for all m.

The hypothesis on nonpositivity of curvature in this theorem is used essentially in our proof. We have also some weaker result without constraining the curvature.

Theorem 8.1.4 *For an Anosov manifold, inclusion (8.1.2) has a finite codimension for every m.*

In the case of an Anosov surface (= two-dimensional Anosov manifold), we have the following analog of Theorem 4.4.1.

Theorem 8.1.5 *For an Anosov surface without focal points, inclusion (8.1.2) is equality for $m = 2$.*

We conclude the section by posing the following question.

Problem 8.1.6 *Does there exist an Anosov manifold such that inclusion (8.1.2) is not equality for some values of m?*

8.2 Spectral rigidity

In the famous lecture by M. Kac [42], the following question was arisen: can one hear the shape and size of a drum? The question is posed more precisely as follows.

Let (M, g) be a closed Riemannian manifold, and $\Delta : C^\infty(M) \to C^\infty(M)$ be the corresponding Laplace — Beltrami operator. Being an elliptic operator, $-\Delta$ has an infinite discrete eigenvalue spectrum $\text{Spec}(M, g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$. Two closed Riemannian manifolds are called *isospectral* if their

eigenvalue spectra coincide. Kac's question can be formulated as follows: do there exist isospectral but not isometric manifolds?

The first example of isospectral manifolds was found by J. Milnor in the dimension 16 [52]. Later M. Vignéras [88] showed that even in the class of closed manifolds of constant negative curvature there are isospectral but not homeomorphic manifolds of any dimension. In order to avoid these examples and linearize the problem, V. Guillemin and D. Kazhdan introduced in [35] the following definition of spectral rigidity.

A smooth one-parameter family g^τ $(-\varepsilon < \tau < \varepsilon)$ of metrics on a closed manifold M is called the *deformation* of a metric g if $g^0 = g$. Such a family is called the *isospectral deformation* if the spectrum of the Laplace — Beltrami operator Δ^τ of the metric g^τ is independent of τ. A deformation g^τ is called the *trivial deformation* if there exists a family φ^τ of diffeomorphisms of M such that $g^\tau = (\varphi^\tau)^* g$. A manifold (M, g) is called *spectrally rigid* if it does not admit a nontrivial isospectral deformation.

Since [35] was published, a number of examples of isospectral deformations of compact manifolds have been given [32, 72]. Hence to rule out isospectral deformations there must be some extra assumption.

For Anosov manifolds, the spectral rigidity problem relates closely to integral geometry. In particular, the following claim is stated in [35].

Theorem 8.2.1 *An Anosov manifold (M, g) is spectrally rigid if inclusion* (8.1.2) *is the equality for $m = 2$.*

This theorem is formulated in [35] in the case of negatively curved manifold. Nevertheless, it is valid for Anosov manifolds too because the proof uses the only fact that the index of any closed geodesic is zero. In fact, Theorem 8.2.1 is a simple corollary of some deep relationship between the eigenvalue spectrum and the singular support of the trace of the wave kernel, established by J. J. Duistermaat and V. Guillemin in [23].

Comparing Theorem 8.2.1 with theorems 8.1.3 and 8.1.5, we obtain the following results.

Theorem 8.2.2 *An Anosov manifold of nonpositive sectional curvature is spectrally rigid.*

Theorem 8.2.3 *An Anosov surface without focal points is spectrally rigid.*

Theorem 8.1.4 says us that, for an Anosov manifold, the space of infinitesimal isospectral deformations has a finite dimension modulo trivial deformations.

For two-dimensional manifolds of negative curvature, Theorem 8.2.2 was proved by V. Guillemin and D. Kazhdan in [35]. The same authors proved this fact for n-dimensional manifolds [36] under a pointwise curvature pinching assumption. That result was later extended by Min-Oo [53] to the case where the curvature operator is negative definite.

A closed Riemannian manifold is said to have a simple length spectrum if there do not exist two different closed geodesics such that the ratio of their lengths is a rational number. This is a generic condition.

Theorem 8.2.4 *Let (M, g) be an Anosov manifold with simple length spectrum, and $\Delta : C^\infty(M) \to C^\infty(M)$ be the corresponding Laplace — Beltrami operator. If real functions $q_1, q_2 \in C^\infty(M)$ are such that the operators $\Delta + q_1$ and $\Delta + q_2$ have coincident eigenvalue spectra, then $q_1 \equiv q_2$.*

This result follows from Theorem 8.1.1 because, under hypotheses of Theorem 8.2.4, eigenvalue spectrum of the operator $\Delta + q$ determines integrals of the potential q over closed geodesics, as is shown in [35].

8.3 Decomposition of a tensor field into the solenoidal and potential parts

Lemma 8.3.1 *Let a complete Riemannian manifold (M, g) be such that there exists an orbit of the geodesic flow which is dense in ΩM. If a symmetric tensor field $v \in C^\infty(S^m \tau'_M)$ satisfies the equation*

$$dv = 0, \qquad (8.3.1)$$

then

 (i) if m is odd, v is identically zero;
 (ii) if $m = 2l$ is even, v is of the form $v = cg^l$, where c is a constant.

Proof. Define the function $\varphi \in C^\infty(TM)$ by the equality $\varphi(x, \xi) = \langle v(x), \xi^m \rangle$. It follows from (8.3.1) and (2.4.18) that φ is constant on every orbit of the geodesic flow. Therefore, the restriction of φ to ΩM is constant. From this, taking the homogeneity of $\varphi(x, \xi)$ in its second argument into account, we obtain

$$\langle v(x), \xi^m \rangle = c|\xi|^m.$$

This equality clearly implies the claim of the lemma.

Theorem 8.3.2 *Let a compact Riemannian manifold (M, g) be such that there exists an orbit of the geodesic flow which is dense in ΩM, and let $k \geq 1$ be an integer.*

1. For even m, every symmetric tensor field $f \in H^k(S^m \tau'_M)$ can be uniquely represented in the form

$$f = dv + {}^s\!f, \tag{8.3.2}$$

where $v \in H^{k+1}(S^{m-1}\tau'_M)$, and the field ${}^s\!f \in H^k(S^m\tau'_M)$ is solenoidal, i.e., satisfies the equation

$$\delta\, {}^s\!f = 0. \tag{8.3.3}$$

These fields satisfy the estimates

$$\|v\|_{k+1} \leq C\|\delta f\|_{k-1}, \qquad \|{}^s\!f\|_k \leq C\|f\|_k \tag{8.3.4}$$

with a constant C independent of f.

2. For odd $m = 2l + 1$, the previous claim is also valid under the additional assumption that v satisfies the relation

$$(v, g^l)_{L_2(S^{2l}\tau'_M)} = 0. \tag{8.3.5}$$

In particular, in both 1 and 2 above if f is smooth then ${}^s\!f$ and v are also smooth.

The terms of decomposition (8.3.2) are called the *potential* and *solenoidal parts* of the symmetric tensor field f respectively.

Proof. Assume existence of symmetric tensor fields v and ${}^s\!f$ satisfying (8.3.2) and (8.3.3), and apply the operator δ to the first of the equalities to obtain

$$\delta dv = \delta f. \tag{8.3.6}$$

Conversely, if equation (8.3.6) has a solution satisfying the first of estimates (8.3.4), then, putting ${}^s\!f = f - dv$, we would arrive at the claim of the theorem.

As we have seen in Section 2.4, the operator

$$\delta d : H^{k+1}(S^{m-1}\tau'_M) \to H^{k-1}(S^{m-1}\tau'_M) \tag{8.3.7}$$

is elliptic. Therefore, its kernel $\mathrm{Ker}(\delta d)$ is a finite-dimensional vector space consisting of smooth fields; the image $\mathrm{Im}(\delta d)$ is a closed subspace in $H^{k-1}(S^{m-1}\tau'_M)$; the orthogonal complement $(\mathrm{Im}(\delta d))^\perp$ is a finite-dimensional vector space consisting of smooth fields; and operator (8.3.7) induces an isomorphism of the topological Hilbert spaces

$$H^{k+1}(S^{m-1}\tau'_M)/\mathrm{Ker}(\delta d) \to \mathrm{Im}(\delta d). \tag{8.3.8}$$

Let us show that

$$\mathrm{Ker}(\delta d) = (\mathrm{Im}(\delta d))^\perp = \{v \in C^\infty(S^{m-1}\tau'_M) \mid dv = 0\}. \tag{8.3.9}$$

Indeed, if $v \in \mathrm{Ker}(\delta d)$, then

$$(dv, dv) = -(v, \delta dv) = 0.$$

If $v \in (\mathrm{Im}(\delta d))^\perp$, then for every $u \in C^\infty(S^{m-1}\tau'_M)$

$$(\delta dv, u) = (v, \delta du) = 0.$$

Therefore, $\delta dv = 0$, i.e., $v \in \mathrm{Ker}(\delta d)$.

Observe that the right-hand side of equation (8.3.6) belongs to $\mathrm{Im}(\delta d)$ since $(\delta f, v) = -(f, dv) = 0$ if $dv = 0$. Therefore, equation (8.3.6) has a solution for every $f \in H^k(S^m\tau'_M)$.

In the case of even m, equalities (8.3.9) with the help of Lemma 8.3.1 imply that $\mathrm{Ker}(\delta d) = 0$. Thus, equation (8.3.6) has a unique solution for every $f \in H^k(S^m\tau'_M)$. Since (8.3.8) is an isomorphism, the first of estimates (8.3.4) holds.

In the case of odd $m = 2l + 1$, equalities (8.3.9) with the help of Lemma 8.3.1 imply that $\mathrm{Ker}(\delta d)$ consists of the fields cg^l. Therefore, equation (8.3.8) has a unique solution satisfying condition (8.3.5). The first of the estimates (8.3.4) also holds for the solution. Thus the theorem is proved.

8.4   The Livčic theorem

In previous lectures we reduced integral geometry problems on a CDRM to inverse problems for the kinetic equation by introducing the function $u(x, \xi)$ with the help of definition (3.5.1). Such a definition is impossible in the case of a closed manifold. Instead of that, we will use the following

Theorem 8.4.1 (the Livčic theorem) *Let $H \in C^\infty(\tau_N)$ be a vector field on a closed manifold N which generates the Anosov flow. If a function $F \in C^\infty(N)$ integrates to zero over every closed orbit of the flow, then there exists a function $u \in C^\infty(N)$ such that $Hu = F$.*

A. N. Livčic [47] constructed the function u and proved that it is Hölder-continuous. Smoothness of the function was proved later [48].

With the help of the Livčic theorem, we will now prove that Theorem 8.1.3 follows from the next claim.

Lemma 8.4.2 *Let (M, g) be a closed non-positively curved Riemannian manifold such that there exists an orbit of the geodesic flow which is dense in ΩM. If a function $u \in C^\infty(\Omega M)$ and a symmetric tensor field $f \in C^\infty(S^m \tau'_M)$ satisfy the equation*

$$Hu(x, \xi) = \langle f(x), \xi^m \rangle \tag{8.4.1}$$

on ΩM, then the field f is potential, i.e., there exists a symmetric tensor field $v \in C^\infty(S^{m-1} \tau'_M)$ such that $dv = f$.

Proof of Theorem 8.1.3. The condition of the theorem means that the integral of the function

$$F(x, \xi) = \langle f(x), \xi^m \rangle$$

over every closed orbit of the geodesic flow is equal to zero. By the Livčic theorem, there exists a function $u \in C^\infty(\Omega M)$ satisfying equation (8.4.1). Applying Lemma 8.4.2, we arrive at the claim of Theorem 8.1.3.

In its turn, Lemma 8.4.2 follows from the following special case.

Lemma 8.4.3 *Let (M, g) be as in Lemma 8.4.2. If a symmetric tensor field $f \in C^\infty(S^m \tau'_M)$ is solenoidal, i.e., satisfies the equation*

$$\delta f = 0, \tag{8.4.2}$$

and if there exists a function $u \in C^\infty(\Omega M)$ satisfying (8.4.1), then $f \equiv 0$.

Proof of Lemma 8.4.2. Let the assumptions of the lemma be fulfilled, and let (8.3.2) be the decomposition of the field f into potential and solenoidal parts. Putting

$$\tilde{u}(x, \xi) = u(x, \xi) - \langle v(x), \xi^{m-1} \rangle$$

from (8.3.2) and (8.4.1) we derive

$$H\tilde{u}(x, \xi) = \langle {}^s f(x), \xi^m \rangle.$$

Assuming Lemma 8.4.3 to be valid, the last equality implies ${}^s f = 0$. Now formula (8.3.2) yields $dv = f$.

Along the same line, we will now show that Theorem 8.1.4 is a corollary of the following

Lemma 8.4.4 *Let (M, g) be an Anosov manifold. If a function $u \in C^\infty(\Omega M)$ and tensor field $f \in C^\infty(S^m \tau'_M)$ are connected by the kinetic equation (8.4.1), then the estimate*

$$\|u\|^2_{H^1(\Omega M)} \leq C \left(\|u\|^2_{L_2(\Omega M)} + \|\delta f\|_{L_2(S^{m-1} \tau'_M)} \cdot \|u\|_{L_2(\Omega M)} \right) \tag{8.4.3}$$

holds with some constant C independent of u and f.

Proof of Theorem 8.1.4. If the theorem is not true, there exists an infinite sequence of tensor fields $z_k \in Z^\infty(S^m \tau'_M)$ $(k = 1, 2, \ldots)$ which is linearly independent $\mathrm{mod}\,(P^\infty(S^m \tau'_M))$. Applying Theorem 8.3.2, we decompose every field z_k into potential and solenoidal parts

$$z_k = y_k + dv_k, \qquad \delta y_k = 0. \tag{8.4.4}$$

Then the sequence $y_k \in Z^\infty(S^m \tau'_M)$ is linearly independent. Applying the Livčic theorem, we find functions $w_k \in C^\infty(\Omega M)$ satisfying the kinetic equation

$$Hw_k = \langle y_k(x), \xi^m \rangle. \tag{8.4.5}$$

The sequence w_k is linearly independent since in the other case (8.4.5) would imply linear dependence of the sequence y_k. Orthogonalizing the sequence w_k, we construct a new sequence of functions $u_k \in C^\infty(\Omega M)$ such that

$$\|u_k\|_{L_2(\Omega M)} = 1, \qquad \|u_k - u_l\|_{L_2(\Omega M)} > 1/2 \quad \text{for} \quad k \neq l, \tag{8.4.6}$$

and every u_k is a linear combination of $w_1, \ldots, w_k$. Equation (8.4.5) implies that

$$Hu_k = \langle f_k(x), \xi^m \rangle,$$

where $f_k \in C^\infty(S^m \tau'_M)$ is a linear combination of $y_1, \ldots, y_k$. Therefore (8.4.4) implies that

$$\delta f_k = 0. \tag{8.4.7}$$

By equalities (8.4.6) and (8.4.7), estimate (8.4.3) has the following form for the pair u_k, f_k:

$$\|u_k\|_{H^1(\Omega M)} \leq C\|u_k\|_{L_2(\Omega M)} = C.$$

In other words, the sequence u_k is bounded in $H^1(\Omega M)$. Since the imbedding $H^1(\Omega M) \subset L_2(\Omega M)$ is compact, the sequence u_k contains a subsequence converging in $L_2(\Omega M)$. But this contradicts to inequality (8.4.6).

8.5   Proof of Lemma 8.4.3

Let f and u satisfy the lemma hypothesis. We extend the function $u(x, \xi)$ onto $T^0 M$ in such a way that the function becomes positively homogeneous of degree $m - 1$ in its second argument. Then $u \in C^\infty(T^0 M)$, and equation (8.4.1) holds on $T^0 M$.

The Pestov identity (4.1.2) holds on $T^0 M$ with the semibasic vector fields v and w defined by (4.1.3) and (4.1.4). The last term on the right-hand side of (4.1.2) is nonnegative because of the hypothesis on sectional curvature. Therefore (4.1.2) implies the inequality

$$|\overset{h}{\nabla}u|^2 \leq 2\langle \overset{h}{\nabla}u, \overset{v}{\nabla}(Hu) \rangle - \overset{h}{\nabla}_i v^i - \overset{v}{\nabla}_i w^i. \tag{8.5.1}$$

Using (8.4.1), we transform the first term in the right-hand side of inequality (8.5.1) as follows

$$2\langle \overset{h}{\nabla}u, \overset{v}{\nabla}(Hu) \rangle = 2\overset{h}{\nabla}^j u \cdot \frac{\partial}{\partial \xi^j}(f_{i_1 \ldots i_m} \xi^{i_1} \ldots \xi^{i_m}) = 2m\overset{h}{\nabla}^j u \cdot f_{ji_2 \ldots i_m} \xi^{i_2} \ldots \xi^{i_m} =$$

$$= \overset{h}{\nabla}^j(2mu f_{ji_2 \ldots i_m} \xi^{i_2} \ldots \xi^{i_m}) - 2mu(\delta f)_{i_2 \ldots i_m} \xi^{i_2} \ldots \xi^{i_m}.$$

Using condition (8.4.2), we obtain

$$2\langle \overset{h}{\nabla}u, \overset{v}{\nabla}(Hu) \rangle = \overset{h}{\nabla}_i \tilde{v}^i, \tag{8.5.2}$$

where

$$\tilde{v}^i = 2mu g^{ij} f_{ji_2 \ldots i_m} \xi^{i_2} \ldots \xi^{i_m}.$$

Replacing the first term on the right-hand side of (8.5.1) by its value (8.5.2), we obtain

$$|\overset{h}{\nabla}u|^2 \leq \overset{h}{\nabla}_i(\tilde{v}^i - v^i) - \overset{v}{\nabla}_i w^i. \tag{8.5.3}$$

We multiply inequality (8.5.3) by the volume form $d\Sigma = d\Sigma^{2n-1}$ and integrate the result over ΩM. Transforming the integrals on the right-hand side of the so-obtained inequality by the Gauss-Ostrogradskiĭ formulas for vertical and horizontal divergences, we arrive at the relation

$$\int_{\Omega M} |\overset{h}{\nabla}u|^2 \, d\Sigma \leq -(n + 2m - 2) \int_{\Omega M} \langle w, \xi \rangle \, d\Sigma. \tag{8.5.4}$$

The constant $n + 2m - 2$ above comes from the fact that the field $w(x, \xi)$ is homogeneous of degree $2m - 1$ in its second argument. Further, since $\langle w, \xi \rangle = |Hu|^2$, inequality (8.5.4) takes the form

$$\int_{\Omega M} |\overset{h}{\nabla}u|^2 \, d\Sigma + (n + 2m - 2) \int_{\Omega M} |Hu|^2 \, d\Sigma \leq 0.$$

Consequently, $Hu \equiv 0$. Now (8.4.1) implies that $f \equiv 0$. The lemma is thus proved.

8.6 Anosov geodesic flows

The stable and unstable distributions of an Anosov manifold allow us to construct two continuous semibasic tensor fields that will be used as modifying tensors in the proof of Lemma 8.4.4.

Lemma 8.6.1 *Let (M, g) be an n-dimensional Anosov manifold. There exist continuous semibasic tensor fields $\overset{s}{\alpha} = (\overset{s}{\alpha}_{ij}(x, \xi)) \in C(\beta_2^0 M; T^0 M)$ and $\overset{u}{\alpha} = (\overset{u}{\alpha}_{ij}(x, \xi)) \in C(\beta_2^0 M; T^0 M)$ defined on $T^0 M$ such that*
(1) the fields are symmetric

$$\overset{s}{\alpha}_{ij} = \overset{s}{\alpha}_{ji}, \quad \overset{u}{\alpha}_{ij} = \overset{u}{\alpha}_{ji}$$

and orthogonal to the vector ξ

$$\xi^i \overset{s}{\alpha}_{ij}(x, \xi) = 0, \quad \xi^i \overset{u}{\alpha}_{ij}(x, \xi) = 0;$$

(2) they are positively homogeneous of degree 1 in ξ

$$\overset{s}{\alpha}(x, t\xi) = t\overset{s}{\alpha}(x, \xi), \quad \overset{u}{\alpha}(x, t\xi) = t\overset{u}{\alpha}(x, \xi) \quad for \quad t > 0;$$

(3) the rank of the matrix $(\overset{s}{\alpha}_{ij} - \overset{u}{\alpha}_{ij})$ equals $n - 1$ at every point $(x, \xi) \in T^0 M$;
(4) along every geodesic $\gamma : \mathbf{R} \to M$, the fields $\overset{s}{\alpha}{}_j^i(t) = (g^{ik}\overset{s}{\alpha}_{kj})(\gamma(t), \dot{\gamma}(t))$ and $\overset{u}{\alpha}{}_j^i(t) = (g^{ik}\overset{u}{\alpha}_{kj})(\gamma(t), \dot{\gamma}(t))$ are smooth and satisfy the Riccati equation

$$\alpha' + \alpha^2 + R = 0, \tag{8.6.1}$$

where the prime denotes the covariant derivative, and $R = R(t)$ is the curvature operator, $R_j^i = R_{kjl}^i \dot{\gamma}^k \dot{\gamma}^l$.

Before proving the lemma we recall some notions concerning the geodesic flow and Jacobi fields.

Let $\tau_M = (TM, p, M)$ be the tangent bundle of a Riemannian manifold (M, g). Given a point $(x, \xi) \in TM$, the canonical isomorphism

$$T_{(x,\xi)}(TM) \cong T_x M \oplus T_x M, \quad v \mapsto (dp(v), Kv)$$

is defined, where $K : TTM \to TM$ is the *connection mapping* [33]. The subspaces of $T_{(x,\xi)}(TM)$ corresponding to the summands of the right-hand side are called the *horizontal* and *vertical spaces* respectively. This isomorphism defines the *Sasakian metric* on TM

$$\langle v, w \rangle = \langle dp(v), dp(w) \rangle + \langle Kv, Kw \rangle.$$

The metric g determines the isomorphism of the tangent and cotangent bundles. The standard symplectic structure of the cotangent bundle, being shifted to TM with the help of the isomorphism, is defined by the 2-form

$$\omega(v, w) = \langle dp(v), Kw \rangle - \langle dp(w), Kv \rangle.$$

The tangent space of the manifold ΩM of unit tangent vectors at a point $(x, \xi) \in \Omega M$ can be distinguished by the equality

$$T_{(x,\xi)}(\Omega M) = \{v \in T_{(x,\xi)}(TM) \mid \langle Kv, \xi \rangle = 0\}. \tag{8.6.2}$$

Let H be the geodesic vector field on TM generating the geodesic flow $G^t : TM \to TM$. Note that H is horizontal, $KH = 0$. Given a vector $v \in T_{(x,\xi)}(TM)$, the vector field $Y_v(t) = dp \circ dG^t(v)$ is a Jacobi vector field along the geodesic $\gamma(t) = \exp_x t\xi$ with the covariant derivative $Y_v'(t) = K \circ dG^t(v)$.

Let now (M, g) be an Anosov manifold. Given $(x, \xi) \in \Omega M$, two $(n - 1)$-dimensional subspaces $X^s(x, \xi)$ and $X^u(x, \xi)$ of $T_{(x,\xi)}(\Omega M)$ are defined which are the stable and unstable subspaces respectively for the geodesic flow. We will use the following properties of these spaces which follow from Proposition 1.7 and Theorem 3.2 of [24].

(i) The distributions $(x, \xi) \mapsto X^s(x, \xi)$ and $(x, \xi) \mapsto X^u(x, \xi)$ are continuous and invariant with respect to the geodesic flow, i.e.,

$$dG^t(X^s(x, \xi)) = X^s(G^t(x, \xi)), \quad dG^t(X^u(x, \xi)) = X^u(G^t(x, \xi)).$$

(ii) Each of the spaces $X^s(x, \xi)$ and $X^u(x, \xi)$ is orthogonal to the vector $H(x, \xi)$. The space $T_{(x,\xi)}(\Omega M)$ splits to the (not orthogonal) direct sum of three subspaces

$$T_{(x,\xi)}(\Omega M) = X^s(x, \xi) \oplus X^u(x, \xi) \oplus \{H(x, \xi)\}.$$

(iii) The restriction of the mapping dp to each of the spaces $X^s(x,\xi)$ and $X^u(x,\xi)$ is an isomorphism onto $N_{(x,\xi)} = \{\eta \in T_x M \mid \langle \xi, \eta \rangle = 0\}$.

(iv) $X^s(x,\xi)$ and $X^u(x,\xi)$ are Lagrangian spaces, i.e., $\omega(v,w) = 0$ for $v,w \in X^s(x,\xi)$ $(X^u(x,\xi))$.

Proof of Lemma 8.6.1. Given $(x,\xi) \in \Omega M$, we define two endomorfisms $b_s(x,\xi)$ and $b_u(x,\xi)$ of the space $N_{(x,\xi)}$ as compositions of the following mappings:

$$b_s(x,\xi): N_{(x,\xi)} \xrightarrow{(dp)^{-1}} X^s(x,\xi) \xrightarrow{K} N_{(x,\xi)},$$

$$b_u(x,\xi): N_{(x,\xi)} \xrightarrow{(dp)^{-1}} X^u(x,\xi) \xrightarrow{K} N_{(x,\xi)}.$$

Note that $Kv \in N_{(x,\xi)}$ for every $v \in T_{(x,\xi)}(\Omega M)$ because of equality (8.6.2). This definition is equivalent to the following rule that is more comfortable for using: two vectors $\eta, \zeta \in N_{(x,\xi)}$ are connected by the equality $b_s(x,\xi)\eta = \zeta$ if and only if there exists $v \in X^s(x,\xi)$ such that $dp(v) = \eta$ and $Kv = \zeta$. The similar rule is valid for the operator $b_u(x,\xi)$.

We establish the following properties of the operators $b_s(x,\xi)$ and $b_u(x,\xi)$.

1. The operators $b_s(x,\xi)$ and $b_u(x,\xi)$ continuously depend on $(x,\xi) \in \Omega M$. If the stable and unstable distributions belong to the class W_p^1, then the functions $(x,\xi) \mapsto b_s(x,\xi)$ and $(x,\xi) \mapsto b_u(x,\xi)$ belong also to W_p^1.

2. The operators b_s and b_u are selfdual. Indeed, let $\eta_i \in N_{(x,\xi)}$ and $\zeta_i = b_s(x,\xi)\eta_i$ $(i = 1,2)$. Then there are $v_i \in X^s(x,\xi)$ such that $dp(v_i) = \eta_i$ and $Kv_i = \zeta_i$. Therefore

$$\langle b_s\eta_1, \eta_2 \rangle - \langle \eta_1, b_s\eta_2 \rangle = \langle \zeta_1, \eta_2 \rangle - \langle \eta_1, \zeta_2 \rangle = \langle Kv_1, dp(v_2) \rangle - \langle dp(v_1), Kv_2 \rangle = \omega(v_1, v_2) = 0$$

since $X^s(x,\xi)$ is a Lagrangian space.

3. The operator $b_s - b_u$ is nondegenerate. Indeed, let $b_s\eta = b_u\eta$ for some vector $\eta \in N_{(x,\xi)}$. There exist vectors $v \in X^s$ and $w \in X^u$ such that

$$dp(v) = \eta = dp(w), \quad Kv = b_s\eta = b_u\eta = Kw.$$

These relations imply that $v = w \in X^s \cap X^u = 0$. Consequently, $v = w = 0$ and $\eta = 0$.

4. Along every unit speed geodesic $\gamma : \mathbf{R} \to M$, each of the operator functions

$$b_s(t) = b_s(\gamma(t), \dot{\gamma}(t)): N_{(\gamma(t), \dot{\gamma}(t))} \to N_{(\gamma(t), \dot{\gamma}(t))},$$

$$b_u(t) = b_u(\gamma(t), \dot{\gamma}(t)): N_{(\gamma(t), \dot{\gamma}(t))} \to N_{(\gamma(t), \dot{\gamma}(t))}$$

satisfies the Riccati equation

$$b' + b^2 + R = 0. \tag{8.6.3}$$

Indeed, fix a geodesic γ and denote $N_t = N_{(\gamma(t), \dot{\gamma}(t))}$. Define the operator function $D(t): N_t \to N_t$ as the solution to the Jacobi equation

$$D'' + RD = 0 \tag{8.6.4}$$

satisfying the initial conditions

$$D(0) = E \ (= \ \text{identity}), \quad D'(0) = b_s(0).$$

Establish validity of the equality

$$D'(t) = b_s(t)D(t). \tag{8.6.5}$$

To this end fix a vector $\eta \in N_0$ and denote by $\eta(t) \in N_t$ the result of parallel translating the vector η along γ. Then the vector function $Y(t) = D(t)\eta(t)$ is the Jacobi vector field along γ satisfying the initial conditions

$$Y(0) = \eta, \quad Y'(0) = b_s(0)\eta.$$

On the other hand, if a vector $v \in X^s(\gamma(0), \dot{\gamma}(0))$ is such that $dp(v) = \eta$, then

$$D(t)\eta(t) = Y(t) = dp \circ dG^t(v), \quad D'(t)\eta(t) = Y'(t) = K \circ dG^t(v).$$

The vector $v_t = dG^t(v)$ belongs to $X^s(\gamma(t), \dot{\gamma}(t))$ because the distribution X^s is invariant with respect to the geodesic flow. The proceeding equalities can be rewritten as follows:

$$D(t)\eta(t) = dp(v_t), \quad D'(t)\eta(t) = Kv_t, \quad v_t \in X^s(\gamma(t), \dot{\gamma}(t)).$$

These relations imply that

$$D'(t)\eta(t) = b_s(t)D(t)\eta(t).$$

The latter equality is equivalent to (8.6.5) because η is an arbitrary vector.

The operator $D(t)$ is nondegenerate for sufficiently small $|t|$, and equality (8.6.5) can be rewritten as follows: $b_s(t) = D'(t)D^{-1}(t)$. From this and Jacobi equation (8.6.4), we see that $b_s(t)$ satisfies Riccati equation (8.6.3) at least for sufficiently small $|t|$. Since the Riccati equation is invariant with respect to the shift $t \mapsto t + t_0$, it is satisfied for all t.

Given $(x, \xi) \in \Omega M$, we define the operators

$$\overset{s}{\alpha}(x, \xi) : T_x M \to T_x M, \quad \overset{u}{\alpha}(x, \xi) : T_x M \to T_x M$$

by the equalities

$$\overset{s}{\alpha}(x, \xi)|_{N_{(x,\xi)}} = b_s(x, \xi), \quad \overset{s}{\alpha}(x, \xi)\xi = 0;$$

$$\overset{u}{\alpha}(x, \xi)|_{N_{(x,\xi)}} = b_u(x, \xi), \quad \overset{u}{a}(x, \xi)\xi = 0.$$

We then extend the functions $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ to $T^0 M$ in such the way that they are positively homogeneous in ξ

$$\overset{s}{\alpha}(x, t\xi) = t\overset{s}{\alpha}(x, \xi), \quad \overset{u}{\alpha}(x, t\xi) = t\overset{u}{\alpha}(x, \xi) \quad \text{for} \quad t > 0.$$

We thus have constructed the semibasic tensor fields $\overset{s}{\alpha} = (\overset{s}{\alpha}{}^i_j(x, \xi)) \in C(\beta^1_1 M; T^0 M)$ and $\overset{u}{\alpha} = (\overset{u}{\alpha}{}^i_j(x, \xi)) \in C(\beta^1_1 M; T^0 M)$. The above proved properties of b_s and b_u imply that the semibasic tensor fields $\overset{s}{\alpha}_{ij} = g_{ik}\overset{s}{\alpha}{}^k_j$ and $\overset{u}{\alpha}_{ij} = g_{ik}\overset{u}{\alpha}{}^k_j$ satisfy all statements of Lemma 8.6.1.

8.7 Smoothing the tensor fields $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$

We would like to use two modified horizontal derivatives defined by the scheme of Section 6.1 with the modifying tensors $\overset{s}{a}$ and $\overset{u}{a}$ constructed in the previous section. Unfortunately, the fields $\overset{s}{a}$ and $\overset{u}{a}$ are only continuous but are not smooth. For constructing a modified horizontal derivative, we need at least C^2-smoothness of the modifying tensor field because the definition of the curvature tensor assumes existence of second order derivatives. Therefore we have to smooth the tensor fields $\overset{s}{a}$ and $\overset{u}{a}$. We will choose the smoothing tensors in such a way that they would satisfy all statements of Lemma 8.6.1 with the following exception: Riccati equation (8.6.1) will be satisfied approximately.

First of all we will discuss some questions concerning smoothing sections of a vector bundle.

Let $\pi : E \to N$ be a smooth m-dimensional vector bundle over a compact manifold N. Choose a finite atlas $\{U_a, \varphi_a\}^A_{a=1}$ of the manifold N, a partition of unity $\{\mu_a\}^A_{a=1}$ subordinated to the atlas, and local trivializations $(e^a_1, \ldots, e^a_m)$ of the bundle over U_a (this means that $e^a_\alpha \in C^\infty(E; U_a)$, and the vectors $e^a_1(x), \ldots, e^a_m(x)$ constitute a basis of the fiber E_x for every point $x \in U_a$). Every section $f \in C(E)$ can be uniquely represented in the form

$$f(x) = \sum_{\alpha=1}^m f^\alpha_a(x)e^a_\alpha(x), \quad x \in U_a. \tag{8.7.1}$$

For $0 \le k < \infty$, let $C^k(E)$ be the space of sections f such that $(\mu_a f^\alpha_a) \circ \varphi^{-1}_a \in C^k(\mathbf{R}^n)$. The norm on the space is defined by the equality

$$\|f\|_{C^k(E)} = \sum_{a=1}^A \sum_{\alpha=1}^m \|(\mu_a f^\alpha_a) \circ \varphi^{-1}_a\|_{C^k(\mathbf{R}^n)}.$$

Up to equivalence, the norm is independent of the choice of the atlas, partition of unity, and trivializations.

Let $H \in C^\infty(\tau_N)$ be a smooth vector field on N. Choosing a connection on E, we can define the derivative Hf of a section f with respect to H. A section $f \in C(E)$ is said *differentiable along H* if the derivative Hf exists and belongs to $C(E)$. In such the case the norm $\|Hf\|_{C(E)}$ is defined and independent, up to equivalence, of the choice of the connection.

Lemma 8.7.1 *Let $H \in C^\infty(\tau_N)$ be a smooth vector field on a compact manifold N which does not vanish at any point, and E be a smooth vector bundle over N. Fix a C-norm on $C(E)$ and a connection on E. For a differentiable along H section $f \in C(E)$ and for $\varepsilon > 0$, there exists a smooth section $\widetilde{f} \in C^\infty(E)$ such that*

$$\|f - \widetilde{f}\|_{C(E)} < \varepsilon, \quad \|H(f - \widetilde{f})\|_{C(E)} < \varepsilon.$$

Proof. A chart (U, φ) of the manifold N is called *straightening the vector field H* if $\varphi_* H$ coincides with the coordinate vector field $\partial / \partial x^1$ on the range $\varphi(U) \subset \mathbf{R}^n$. There exists a finite atlas $\{U_a, \varphi_a\}_{a=1}^A$ consisting of straightening charts [13]. Choose a partition of unity subordinated to the atlas and trivializations of E over U_a. Given a differentiable along H section $f \in C(E)$, representation (8.7.1) holds where, for every a and α, $\widetilde{f}_a^\alpha = (\mu_a f^\alpha) \circ \varphi_a^{-1}$ is a continuous compactly supported function on $\mathbf{R}^n$ with the continuous derivative $\partial \widetilde{f}_a^\alpha / \partial x^1$.

Fix a nonnegative function $\lambda \in C_0^\infty(\mathbf{R}^n)$ such that $\int_{\mathbf{R}^n} \lambda \, dx = 1$, and put $\lambda_\delta(x) = \lambda(x/\delta)/\delta^n$ for $\delta > 0$. For every indices a and α, the function $\overset{\delta}{f}_a^\alpha = \widetilde{f}_a^\alpha * \lambda_\delta$ is C^∞-smooth on $\mathbf{R}^n$, and $\operatorname{supp} \overset{\delta}{f}_a^\alpha \subset \varphi_a(U_a)$ for sufficiently small δ. The differences $\overset{\delta}{f}_a^\alpha - \widetilde{f}_a^\alpha$ and $\partial(\overset{\delta}{f}_a^\alpha - \widetilde{f}_a^\alpha)/\partial x^1$ tend to zero uniformly on $\mathbf{R}^n$ as $\delta \to 0$. Therefore the section

$$\widetilde{f} = \sum_{a=1}^A \sum_{\alpha=1}^m \mu_a (\overset{\delta}{f}_a^\alpha \circ \varphi_a) e_\alpha^a$$

possesses all the desired properties for sufficiently small $\delta > 0$. The lemma is proved.

Given an Anosov manifold (M, g), let $\beta_2^0 M|_{\Omega M}$ be the restriction of the bundle $\beta_2^0 M$ to the compact manifold $N = \Omega M$, and E be its subbundle consisting of semibasic tensors $f = (f_{ij})$ satisfying the conditions $f_{ij} = f_{ji}$ and $\xi^i f_{ij} = 0$. Let $\overset{s}{\alpha}, \overset{u}{\alpha} \in C(E)$ be the sections constructed in Lemma 8.6.1, and H be the geodesic vector field on ΩM. Applying Lemma 8.7.1 to $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ and extending the so-obtained smooth fields to $T^0 M$ by homogeneity, we obtain the following

Lemma 8.7.2 *Let (M, g) be an n-dimensional Anosov manifold. For every $\varepsilon > 0$, there exist smooth semibasic tensor fields $\overset{s}{a} = (\overset{s}{a}_{ij}(x, \xi)) \in C^\infty(\beta_2^0 M; T^0 M)$ and $\overset{u}{a} = (\overset{u}{a}_{ij}(x, \xi)) \in C^\infty(\beta_2^0 M; T^0 M)$ such that*

(1) the fields are symmetric:

$$\overset{s}{a}_{ij} = \overset{s}{a}_{ji}, \quad \overset{u}{a}_{ij} = \overset{u}{a}_{ji}$$

and orthogonal to the vector ξ:

$$\xi^i \overset{u}{a}_{ij}(x, \xi) = 0, \quad \xi^i \overset{u}{a}_{ij})x, \xi) = 0;$$

(2) they are positively homogeneous of degree 1 in ξ:

$$\overset{s}{a}(x, t\xi) = t\overset{s}{a}(x, \xi), \quad \overset{u}{a}(x, t\xi) = t\overset{u}{a}(x, \xi) \quad \text{for} \quad t > 0;$$

(3) the rank of the matrix $(\overset{s}{a}_{ij} - \overset{u}{a}_{ij})$ equals $n - 1$ at every point $(x, \xi) \in \Omega M$;

(4) along every unit speed geodesic $\gamma : \mathbf{R} \to M$, the fields $\overset{s}{a}_j^i(t) = (g^{ik} \overset{s}{a}_{kj})(\gamma(t), \dot{\gamma}(t))$ and $\overset{u}{a}_j^i(t) = (g^{ik} \overset{u}{a}_{kj})(\gamma(t), \dot{\gamma}(t))$ satisfy the inequality

$$|a' + a^2 + R| < \varepsilon$$

for all $t \in \mathbf{R}$.

8.8   The modified horizontal derivatives $\overset{s}{\nabla}$ and $\overset{u}{\nabla}$

Let now (M, g) be an Anosov manifold, and $\overset{s}{a}$ $(\overset{u}{a})$ be the semibasic tensor field constructed in Lemma 8.7.2. Setting $a = \overset{s}{a}$ $(\overset{u}{a})$ in (6.1.7), we define the modified horizontal derivative on $C^\infty(\beta_s^r M; T^0 M)$ which will be denoted by $\overset{s}{\nabla}$ $(\overset{u}{\nabla})$. The corresponding curvature tensor will be denoted by $\overset{s}{R}$ $(\overset{u}{R})$.

Comparing (6.1.24) with statement (4) of Lemma 8.7.2, we arrive at the following important conclusion: for every semibasic vector fields $v, w \in C^\infty(\beta_0^1 M)$, the estimates

$$|\overset{s}{R}_{ijkl} \, \xi^i \xi^k v^j w^l| < \varepsilon |v||w|, \quad |\overset{u}{R}_{ijkl} \, \xi^i \xi^k v^j w^l| < \varepsilon |v||w| \tag{8.8.1}$$

hold uniformly on ΩM.

By statement (3) of Lemma 8.7.2, the set

$$\overset{s}{\nabla}^i u, \quad \overset{u}{\nabla}^i u \ (i = 1, \dots, n), \quad \xi^k \overset{v}{\nabla}_k u$$

is a full family of derivatives of a function $u \in C^\infty(T^0 M)$, i.e., every first-order derivative of u is a linear combination of the set. This observation is specified by the following

Lemma 8.8.1 *For every function $u \in C^\infty(T^0M)$, the estimates*

$$|\overset{v}{\nabla}u - \langle \xi, \overset{v}{\nabla}u \rangle \xi| \leq C(|\overset{s}{\nabla}u| + |\overset{u}{\nabla}u|), \tag{8.8.2}$$

$$|\overset{h}{\nabla}u| \leq C(|\overset{s}{\nabla}u| + |\overset{u}{\nabla}u|) \tag{8.8.3}$$

hold on ΩM with some constant C independent of u.

Proof. It suffices to consider a function u whose support is contained in the domain $U \subset T^0M$ of a local coordinate system. The semibasic vector field $y = \overset{v}{\nabla}u - \langle \xi, \overset{v}{\nabla}u \rangle \xi$ is orthogonal to ξ on ΩM. By the definition of the modified derivatives

$$\overset{s}{\nabla}^i u = \overset{h}{\nabla}^i u + \overset{s}{a}^{ij}\overset{v}{\nabla}_j u, \quad \overset{u}{\nabla}^i u = \overset{h}{\nabla}^i u + \overset{u}{a}^{ij}\overset{v}{\nabla}_j u. \tag{8.8.4}$$

Substituting $\overset{v}{\nabla}_j u = y_j + \langle \xi, \overset{v}{\nabla}u \rangle \xi_j$ into these equalities and using orthogonality of $\overset{s}{a}$ and $\overset{u}{a}$ to ξ, we obtain

$$\overset{s}{\nabla}^i u = \overset{h}{\nabla}^i u + \overset{s}{a}^{ij}y_j, \quad \overset{u}{\nabla}^i u = \overset{h}{\nabla}^i u + \overset{u}{a}^{ij}y_j.$$

This implies that

$$(\overset{s}{a}^{ij} - \overset{u}{a}^{ij})y_j = \overset{s}{\nabla}^i u - \overset{u}{\nabla}^i u. \tag{8.8.5}$$

By statements (1) and (3) of Lemma 8.7.2, the operator $\overset{s}{a} - \overset{u}{a}$ is an automorphism of the space $N_{(x,\xi)} = \{\eta \in T_xM \mid \langle \xi, \eta \rangle = 0\}$. The right-hand side of (8.8.5) belongs to this space because $\langle \xi, \overset{s}{\nabla}u - \overset{u}{\nabla}u \rangle = \xi_i \overset{s}{\nabla}^i u - \xi_i \overset{u}{\nabla}^i u = Hu - Hu = 0$. Therefore equation (8.8.5) has a unique solution

$$y_i = \alpha_{ij}(\overset{s}{\nabla}^j u - \overset{u}{\nabla}^j u)$$

with some coefficients α_{ij} that are smooth functions in U and are independent of u. From this we obtain (8.8.2). The estimate (8.8.3) follows from (8.8.2) and (8.8.4). The lemma is proved.

For a semibasic tensor field $u \in C^\infty(\beta_s^r M; T^0M)$, we will use the notations

$$\|u\|_{L_2}^2 = \int\limits_{\Omega M} |u(x,\xi)|^2 \, d\Sigma(x,\xi), \quad \|u\|_{H^1}^2 = \|u\|_{L_2}^2 + \|\overset{h}{\nabla}u\|_{L_2}^2 + \|\overset{v}{\nabla}u\|_{L_2}^2,$$

where $d\Sigma(x,\xi) = d\omega_x(\xi) \wedge dV^n(x)$ is the volume form on ΩM. Lemma 8.8.1 has the following

Corollary 8.8.2 *The two norms $\|u\|_{H^1}$ and $(\|\overset{s}{\nabla}u\|_{L_2}^2 + \|\overset{u}{\nabla}u\|_{L_2}^2 + \|u\|_{L_2}^2)^{1/2}$ are equivalent on the space of positively homogeneous in ξ functions $u(x,\xi) \in C^\infty(T^0M)$ with the same degree of homogeneity.*

Indeed, $\langle \xi, \overset{v}{\nabla}u \rangle = \lambda u$ if u is homogeneous of degree λ.

Remark. The numbers ε and C participating in (8.8.1)–(8.8.3) are independent in the following sense: ε can be chosen arbitrary small with a fixed value of C. Indeed, C is determined by the continuous fields $\overset{s}{a}$ and $\overset{u}{a}$ constructed in Lemma 8.6.1, while ε depends on the degree of approximating these fields by smooth ones.

8.9 Proof of Lemma 8.4.4

Let a tensor field $f \in C^\infty(S^m\tau_M')$ and a function $u \in C^\infty(\Omega M)$ satisfy equation (8.4.1). We extend the function u to T^0M in such the way that it is positively homogeneous of degree $m-1$ in ξ. Then equation (8.4.1) holds on T^0M. By (6.1.12), this equation can be rewritten in the form

$$Hu = \xi_k\overset{s}{\nabla}^k u = \xi_k\overset{u}{\nabla}^k u = \langle f(x), \xi^m \rangle. \tag{8.9.1}$$

For the modified horizontal derivative $\overset{s}{\nabla}$, the Pestov identity (6.1.28) takes the form:

$$2\langle \overset{s}{\nabla}u, H\overset{v}{\nabla}u \rangle = |\overset{s}{\nabla}u|^2 + \overset{s}{\nabla}^i v_i + \overset{v}{\nabla}_i w^i - \overset{s}{R}_{ijkl}\, \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u, \tag{8.9.2}$$

where

$$v_i = \xi_i \overset{s}{\nabla}^j u \cdot \overset{v}{\nabla}_j u - \xi_j \overset{v}{\nabla}_i u \cdot \overset{s}{\nabla}^j u, \tag{8.9.3}$$

$$w^i = \xi_j \overset{s}{\nabla}^i u \cdot \overset{s}{\nabla}^j u. \tag{8.9.4}$$

We transform the left-hand side of identity (8.9.2). From (8.9.1) we obtain

$$\overset{v}{\nabla}_k(Hu) = \overset{v}{\nabla}_k(\langle f(x), \xi^m \rangle) = m f_{ki_2...i_m} \xi^{i_2} \ldots \xi^{i_m}.$$

Since $\overset{s}{\nabla}^i \xi_j = 0$, this implies

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu \rangle = 2m \overset{s}{\nabla}^i u \cdot f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m} = \overset{s}{\nabla}^i(2mu f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m}) - 2mu \overset{s}{\nabla}^i(f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m}).$$

Introducing the notation

$$\widetilde{v}_i = 2mu f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m},$$

we have

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu \rangle = \overset{s}{\nabla}^i \widetilde{v}_i - 2mu \overset{s}{\nabla}^i(f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m}). \tag{8.9.5}$$

By the definition of the modified derivative,

$$\overset{s}{\nabla}^i(f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m}) = \overset{h}{\nabla}^i(f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m}) +$$

$$+ \overset{s}{a}^{ip} \overset{v}{\nabla}_p(f_{ii_2...i_m} \xi^{i_2} \ldots \xi^{i_m}) + \overset{v}{\nabla}_i \overset{s}{a}^{ip} \cdot (f_{pi_2...i_m} \xi^{i_2} \ldots \xi^{i_m}) =$$

$$= (\delta f)_{i_2...i_m} \xi^{i_2} \ldots \xi^{i_m} + (m-1) \overset{s}{a}^{ip} f_{ipi_3...i_m} \xi^{i_3} \ldots \xi^{i_m} + \overset{v}{\nabla}_i \overset{s}{a}^{ip} \cdot f_{pi_2...i_m} \xi^{i_2} \ldots \xi^{i_m}.$$

Substituting this expression into (8.9.5), we obtain

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu \rangle = \overset{s}{\nabla}^i \widetilde{v}_i - 2mu(\delta f)_{i_2...i_m} \xi^{i_2} \ldots \xi^{i_m} -$$

$$- 2m(m-1) \overset{s}{a}^{ip} u f_{ipi_3...i_m} \xi^{i_3} \ldots \xi^{i_m} - 2m \overset{v}{\nabla}_i \overset{s}{a}^{ip} \cdot u f_{pi_2...i_m} \xi^{i_2} \ldots \xi^{i_m}.$$

With the help of the latter formula, we transform Pestov identity (8.9.2) to the form

$$|\overset{s}{\nabla} u|^2 = \overset{s}{\nabla}^i(\widetilde{v}_i - v_i) - \overset{v}{\nabla}_i w^i - 2mu(\delta f)_{i_2...i_m} \xi^{i_2} \ldots \xi^{i_m} - 2m(m-1) \overset{s}{a}^{ip} u f_{ipi_3...i_m} \xi^{i_3} \ldots \xi^{i_m}$$

$$- 2m \overset{v}{\nabla}_i \overset{s}{a}^{ip} \cdot u f_{pi_2...i_m} \xi^{i_2} \ldots \xi^{i_m} + \overset{s}{R}_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u. \tag{8.9.6}$$

With the help of statement (4) of Lemma 8.7.2 and inequality (8.8.1), the last tree terms on the right-hand side of (8.9.6) can be estimated at a point $(x, \xi) \in \Omega M$ as follows:

$$|\overset{s}{a}^{ip} u f_{ipi_3...i_m} \xi^{i_3} \ldots \xi^{i_m}| \leq C|u||f|,$$

$$|\overset{v}{\nabla}_i \overset{s}{a}^{ip} \cdot u f_{pi_2...i_m} \xi^{i_2} \ldots \xi^{i_m}| \leq |\overset{v}{\nabla} \overset{s}{a}||u||f|,$$

$$|\overset{s}{R}_{ijkl} \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u| < \varepsilon |\overset{v}{\nabla} u|^2$$

with $C = \|\overset{s}{a}\|_{C^1}$. With the help of these estimates, (8.9.6) gives the inequality

$$|\overset{s}{\nabla} u|^2 \leq \overset{s}{\nabla}^i(\widetilde{v}_i - v_i) - \overset{v}{\nabla}_i w^i + C|u||f| + 2m|u||\delta f| + \varepsilon |\overset{v}{\nabla} u|^2 + 2m|\overset{v}{\nabla} \overset{s}{a}||u||f| \tag{8.9.7}$$

that is valid on ΩM with some new constant C.

We integrate inequality (8.9.7) over ΩM and transform the integrals of divergent terms by the Gauss—Ostrogradskiĭ formulas (Theorem 2.7.1 and formula (6.1.32)) The integral of $\overset{s}{\nabla}^i(\widetilde{v}_i - v_i)$ equals to zero because ΩM is a closed manifold. The integral of the second term is nonpositive; indeed

$$- \int_{\Omega M} \overset{v}{\nabla}_i w^i \, d\Sigma = -(n + 2m - 2) \int_{\Omega M} \langle w, \xi \rangle \, d\Sigma = -(n + 2m - 2) \int_{\Omega M} |Hu|^2 \, d\Sigma \leq 0.$$

The latter equality is written because $\langle w, \xi \rangle = |Hu|^2$ as is seen from (8.9.4). Thus, after integration (8.9.7) gives us the inequality

$$\|\overset{s}{\nabla} u\|_{L_2}^2 \leq C(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2}) + \varepsilon \|u\|_{H^1}^2. \tag{8.9.8}$$

We can repeat our arguments with $\overset{u}{\nabla}$ in place of $\overset{s}{\nabla}$. In such the way we obtain the following analog of (8.9.8):

$$\|\overset{u}{\nabla}u\|_{L_2}^2 \leq C(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2}) + \varepsilon\|u\|_{H^1}^2. \tag{8.9.9}$$

With the help of Corollary 8.8.2, inequalities (8.9.8) and (8.9.9) give

$$\|u\|_{H^1}^2 \leq CC'(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2}) + C'\|u\|_{L_2}^2 + C'\varepsilon\|u\|_{H^1}^2. \tag{8.9.10}$$

As we have emphasized, the number ε can be chosen arbitrary small with some fixed value of C'. In particular, we can assume that $C'\varepsilon < 1$, and inequality (8.9.10) can be rewritten in the form

$$\|u\|_{H^1}^2 \leq C(\|u\|_{L_2} \cdot \|f\|_{L_2} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2} + \|u\|_{L_2}^2) \tag{8.9.11}$$

with some new constant C independent of u.

The kinetic equation $Hu = \xi^i \overset{h}{\nabla}_i u = \langle f, \xi^m \rangle$ implies the estimate $\|f\|_{L_2} \leq C\|\overset{h}{\nabla}u\|_{L_2} \leq C\|u\|_{H^1}$ that allows us to rewrite (8.9.11) in the form

$$\|u\|_{H^1}^2 \leq C(\|u\|_{L_2} \cdot \|u\|_{H^1} + \|u\|_{L_2} \cdot \|\delta f\|_{L_2} + \|u\|_{L_2}^2). \tag{8.9.12}$$

Considering (8.9.12) as a quadratic inequality in $x = \|u\|_{H^1}$, one can easily see that it implies (8.4.3) with another constant C. The lemma is proved.

Proof of Theorem 8.1.1 Let a function $f \in C^\infty(M)$ integrates to zero over every closed geodesic. Applying the Livčic theorem, we obtain the function $u \in C^\infty(\Omega M)$ satisfying the kinetic equation

$$Hu(x,\xi) = f(x) \tag{8.9.13}$$

on ΩM. Extending u to $T^0 M$ in such the way as $u(x, t\xi) = t^{-1}u(x,\xi)$ for $t > 0$, equation (8.9.13) is satisfied on $T^0 M$. The left-hand side of the Pestov identity (8.9.2) is identical zero in our case. After integration over ΩM, the identity gives

$$\|\overset{s}{\nabla}u\|_{L_2}^2 + (n-2)\|Hu\|_{L_2}^2 = \int\limits_{\Omega M} \overset{s}{R}_{ijkl}\, \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u \, d\Sigma. \tag{8.9.14}$$

Let $y^i = \overset{v}{\nabla}^i u - \langle \xi, \overset{v}{\nabla}u \rangle \xi^i$. Then, using the symmetries of the curvature tensor, we obtain

$$\overset{s}{R}_{ijkl}\, \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u = \overset{s}{R}_{ijkl}\, \xi^i \xi^k y^j y^l.$$

With the help of (8.8.1) and (8.8.2), the latter equality implies the estimate

$$|\,\overset{s}{R}_{ijkl}\, \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u| \leq \varepsilon|y|^2 \leq C\varepsilon(|\overset{s}{\nabla}u|^2 + |\overset{u}{\nabla}u|^2).$$

Combining this estimate with (8.9.14), we obtain

$$\|\overset{s}{\nabla}u\|_{L_2}^2 \leq C\varepsilon(\|\overset{s}{\nabla}u\|_{L_2}^2 + \|\overset{u}{\nabla}u\|_{L_2}^2). \tag{8.9.15}$$

In the same way we obtain the similar estimate

$$\|\overset{u}{\nabla}u\|_{L_2}^2 \leq C\varepsilon(\|\overset{s}{\nabla}u\|_{L_2}^2 + \|\overset{u}{\nabla}u\|_{L_2}^2). \tag{8.9.16}$$

If $C\varepsilon < 1/2$, inequalities (8.9.15) and (8.9.16) imply that $\overset{s}{\nabla}u = \overset{u}{\nabla}u = 0$. Consequently, $f = \xi_i \overset{s}{\nabla}^i u = 0$. The theorem is proved.

Proof of Theorem 8.1.2. (Compare with the proof of Theorem 6.3.2 in Section 6.5.) In this case the kinetic equation looks as follows:

$$Hu(x,\xi) = f_i(x)\xi^i, \tag{8.9.17}$$

and $\overset{v}{\nabla}Hu = f$. Therefore the Pestov identity (8.9.2) has the form

$$2\langle\overset{s}{\nabla}u, f\rangle = |\overset{s}{\nabla}u|^2 + \overset{s}{\nabla}^i v_i + \overset{v}{\nabla}_i w^i - \overset{s}{R}_{ijkl}\, \xi^i \xi^k \overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u.$$

After integration over ΩM this gives

$$\|\overset{s}{\nabla}u\|_{L_2}^2 - 2(\overset{s}{\nabla}u, f)_{L_2} + n\|Hu\|_{L_2}^2 = \int\limits_{\Omega M} \overset{s}{R}_{ijkl}\,\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u\, d\Sigma. \tag{8.9.18}$$

From (8.9.17), we obtain

$$\|Hu\|_{L_2}^2 = \int\limits_{\Omega M} f_i(x)f_j(x)\xi^i\xi^j\,d\Sigma(x,\xi) = \int\limits_{M} f_i(x)f_j(x)\left[\int\limits_{\Omega_x M} \xi^i\xi^j\,d\omega_x(\xi)\right]dV^n(x) = \frac{1}{n}\|f\|_{L_2}^2.$$

With the help of the latter equality, (8.9.18) takes the form

$$\|\overset{s}{\nabla}u - f\|_{L_2}^2 = \int\limits_{\Omega M} \overset{s}{R}_{ijkl}\,\xi^i\xi^k\overset{v}{\nabla}^j u \cdot \overset{v}{\nabla}^l u\, d\Sigma.$$

Estimating the right-hand side integral with the help of (8.8.1), we obtain

$$\|\overset{s}{\nabla}u - f\|_{L_2}^2 \le \varepsilon C\|\overset{v}{\nabla}u\|_{L_2}^2. \tag{8.9.19}$$

Repeating our arguments with interchanged $\overset{s}{\nabla}$ and $\overset{u}{\nabla}$, we obtain the similar inequality

$$\|\overset{u}{\nabla}u - f\|_{L_2}^2 \le \varepsilon C\|\overset{v}{\nabla}u\|_{L_2}^2. \tag{8.9.20}$$

Let now ε tend to zero in (8.9.19) and (8.9.20). The vector fields f and $\overset{v}{\nabla}u$ are independent of ε as well as the constant C. The fields $\overset{s}{\nabla}u$ and $\overset{u}{\nabla}u$ tend respectively to

$$\overset{s}{\nabla}_i u = \overset{h}{\nabla}_i u + \overset{s}{\alpha}{}_i^p\overset{v}{\nabla}_p u, \quad \overset{u}{\nabla}_i u = \overset{h}{\nabla}_i u + \overset{u}{\alpha}{}_i^p\overset{v}{\nabla}_p u \tag{8.9.21}$$

with the continuous tensors $\overset{s}{\alpha}$ and $\overset{u}{\alpha}$ constructed in Lemma 8.6.1. Thus, passing to limit in (8.9.19) and (8.9.20) as $\varepsilon \to 0$, we obtain

$$\overset{s}{\nabla}_i u = f_i = \overset{u}{\nabla}_i u. \tag{8.9.22}$$

Now equalities (8.9.21) give us

$$(\overset{s}{\alpha}{}_i^p - \overset{u}{\alpha}{}_i^p)\overset{v}{\nabla}_p u = 0. \tag{8.9.23}$$

In our case the function $u(x,\xi)$ is positively homogeneous of zero degree, and therefore

$$\xi^k\overset{v}{\nabla}_k u = 0. \tag{8.9.24}$$

With the help of statement (3) of Lemma 8.6.1, (8.9.23) and (8.9.24) imply that $\overset{v}{\nabla}u = 0$, i.e., the function u is independent of ξ; $u = u(x)$. Equalities (8.9.21) and (8.9.22) take now the form $f_i = \overset{h}{\nabla}_i u = \partial u/\partial x^i$. Therefore f is the exact form, $f = du$. The theorem is proved.

8.10 Proof of Theorem 8.1.5

First of all we will reduce the question to the case of orientable M. Let M be a nonorientable Riemannian surface satisfying the hypotheses of Theorem 8.1.5, and $\pi : \widetilde{M} \to M$ be the twofold covering with the orientable $\widetilde{M}$. Then $\widetilde{M}$ satisfies also the hypotheses of the theorem. In particular, almost every unit speed geodesic of $\widetilde{M}$ is dense in $\Omega\widetilde{M}$. Let $\eta : \widetilde{M} \to \widetilde{M}$ be the isometry changing two points in every fiber of π. By $\eta^* : C^\infty(S^k\tau'_{\widetilde{M}}) \to C^\infty(S^k\tau'_{\widetilde{M}})$ and $\pi^* : C^\infty(S^k\tau'_M) \to C^\infty(S^k\tau'_{\widetilde{M}})$ we denote the mappings of tensor fields induced by η and π respectively. Since η is an isometry, η^* commutes with the operator d of inner differentiation.

Given $f \in Z^\infty(S^2\tau'_M)$, the field $\widetilde{f} = \pi^*f$ belongs to $Z^\infty(S^2\tau'_{\widetilde{M}})$ and satisfies the relation $\eta^*\widetilde{f} = \widetilde{f}$. Assuming Theorem 8.1.5 to be valid for $\widetilde{M}$, we find $\widetilde{v} \in C^\infty(\tau'_{\widetilde{M}})$ such that $\widetilde{f} = d\widetilde{v}$. We have to prove that $\widetilde{v}$ is the lifting of some covector field v on M, i.e., that $\widetilde{v} = \pi^*v$. To this end we should demonstrate that $\eta^*\widetilde{v} = \widetilde{v}$. Indeed,

$$d(\eta^*\widetilde{v} - \widetilde{v}) = \eta^*(d\widetilde{v}) - d\widetilde{v} = \eta^*\widetilde{f} - \widetilde{f} = 0.$$

We have thus shown that
$$d(\eta^*\widetilde{v} - \widetilde{v}) = 0.$$
With the help of Lemma 8.3.1, the latter equality implies that
$$\eta^*\widetilde{v} - \widetilde{v} = 0.$$

In section 8.4, we have proved that Theorem 8.1.3 can be reduced to Lemma 8.4.2. Along the same line, Theorem 8.1.5 follows from the next claim.

Lemma 8.10.1 *Let (M, g) be an orientable Anosov surface without focal points. If a function $u \in C^\infty(\Omega M)$ and a symmetric tensor field $f \in C^\infty(S^2\tau'_M)$ satisfy the equation*
$$Hu(x, \xi) = \langle f(x), \xi^2 \rangle \tag{8.10.1}$$
on ΩM, then the field f is potential, i.e., there exists a covector field $v \in C^\infty(\tau'_M)$ such that $dv = f$.

The following statement is an analog of Lemma 4.4.4.

Lemma 8.10.2 *Under hypotheses of Theorem 8.1.5, there exists a function $b \in C(\Omega M)$ which is smooth on every orbit of the geodesic flow and satisfies the inequality*
$$Hb + 2b^2 + K \leq 0, \tag{8.10.2}$$
where K is the Gaussian curvature.

Proof. In fact we will repeat the arguments of E. Hopf [40] with a slight modification related to absence of focal points.

We consider the Jacobi equation
$$\ddot{y} + Ky = 0 \tag{8.10.3}$$
along a unit speed geodesic $\gamma : \mathbf{R} \to \Omega M$. Absence of conjugate points means that every nontrivial solution to the equation has at most one zero, and any two solutions coincide at most at one point if they are not equal identically. For $a \neq b$, let $y(t; a, b)$ be the solution satisfying the conditions
$$y(a; a, b) = 0, \quad y(b; a, b) = 1. \tag{8.10.4}$$
These functions satisfy the identity
$$y(t; a, b) = y(\beta; a, b)y(t; \alpha, \beta) + y(\alpha; a, b)y(t; \beta, \alpha). \tag{8.10.5}$$
Indeed, the both sides of the equality are solutions to (8.10.5) which, by (8.10.4), coincide at $t = \alpha, \beta$ and therefore are equal identically. In the particular case of $\alpha = a$ and $\beta = b'$, (8.10.5) gives
$$y(t; a, b) = y(b'; a, b)y(t; a, b'). \tag{8.10.6}$$
Since γ has no focal points,
$$\dot{y}(t; a, b) > 0 \quad \text{for} \quad a < b \quad \text{and every } t, \tag{8.10.7}$$
and
$$y(t; a, b) < 0 \quad \text{for} \quad t < a < b, \qquad y(t; a, b) > 0 \quad \text{for} \quad a < b \quad \text{and} \quad t > a. \tag{8.10.8}$$
Two solutions $y(t; a, b)$ and $y(t; a', b)$, $a' < a$, coincide at $t = b$ and at no other point. Consequently, by (8.10.8),
$$y(t; a', b) \leq y(t; a, b) \quad \text{for} \quad a' < a < b \leq t. \tag{8.10.9}$$
(8.10.8) and (8.10.9) imply existence of the limit
$$y(t; b) = \lim_{a \to -\infty} y(t; a, b) \tag{8.10.10}$$
for every $t \geq b$.

If α and β in (8.10.5) are chosen more than b, it becomes evident that limit (8.10.8) exists for every t, and $y(t; b)$ is a solution to equation (8.10.3). This implies also that
$$\dot{y}(t; b) = \lim_{a \to -\infty} \dot{y}(t; a, b)$$

for every t. From (8.10.4), (8.10.7) and (8.10.8) we obtain that

$$y(b; b) = 1, \quad y(t; b) \geq 0, \quad \dot{y}(t; b) \geq 0$$

for all t. Since $y(t; b)$ is a solution to (8.10.3), we have the strong inequality $y(t; b) > 0$ for all t.

The function

$$u(t) = \frac{\dot{y}(t; b)}{y(t; b)}$$

is independent of b by (8.10.8). This function is nonnegative, depends smoothly on t, and satisfies the Riccati equation

$$\dot{u} + u^2 + K = 0.$$

We have thus constructed the function $u(t)$ for every unit speed geodesic $\gamma : \mathbf{R} \to \Omega M$. The value $u(t)$ depends only on the point $\gamma(t)$ but not on the choice of the origin $\gamma(0)$ on γ. In other words, u can be considered as a well-defined function $u(x, \xi)$ on ΩM. As E. Hopf has mentioned in [40] without proof, the function u is continuous on ΩM. In our case continuity of u can be justified as follows. As we have seen in Section 8.6, there is a one-to-one correspondence between the function u and the stable distribution in the case of an Anosov geodesic flow. This means that the stable distribution can be expressed in terms of u and vise versa. On the other hand, it is well known [11] that the stable distribution of an Anosov flow is continuous. This implies continuity of u.

We have thus defined a nonnegative function $u \in C(\Omega M)$ which is smooth on orbits of the geodesic flow and satisfies the Riccati equation

$$Hu + u^2 + K = 0.$$

We define now a new function $a \in C(\Omega M)$ by putting

$$a(x, \xi) = u(x, \xi) - u(x, -\xi).$$

It satisfies the equation

$$Ha + a^2 + 2K = -2u(x, \xi)u(x, -\xi).$$

Since u is nonnegative, a satisfies the inequality

$$Ha + a^2 + 2K \leq 0.$$

Finally, putting $a = 2b$, we arrive at (8.10.2). The lemma is proved.

Proof of Lemma 8.10.1. Let $f \in C^\infty(S^2\tau'_M)$ and $u \in C^\infty(\Omega M)$ satisfy the kinetic equation (8.10.1).

Fixing an orientation on M, the differential operator $\partial/\partial\theta$ is well-defined on ΩM. Our aim is to prove that $\varphi = u_{\theta\theta} + u$ is a constant function. Fix an arbitrary function $c \in C^\infty(\Omega M)$.

There exists a system of isothermic coordinates (x, y) in a neighborhood of every point of M such that the metric has form (4.4.1) in these coordinates. In the domain of isothermic coordinates, we can define the function $a = e^\mu c$ and write down the identity (4.4.8) for the function φ:

$$-4(L^\perp\varphi - \frac{1}{2}a\varphi_\theta)^2 = [\varphi_x + (\mu_y - a\sin\theta)\varphi_\theta]^2 + [\varphi_y + (-\mu_x + a\cos\theta)\varphi_\theta]^2 +$$

$$+ \frac{\partial}{\partial x}\left[\varphi_y\varphi_\theta + (-\mu_x + a\cos\theta)\varphi_\theta^2\right] - \frac{\partial}{\partial y}\left[\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_\theta^2\right] +$$

$$+ \frac{\partial}{\partial\theta}\left[L^\perp\varphi \cdot L\varphi + (\mu_x - a\cos\theta)\varphi_x\varphi_\theta + (\mu_y - a\sin\theta)\varphi_y\varphi_\theta\right] -$$

$$- e^{2\mu}\left[H(e^{-\mu}a) + 2e^{-2\mu}a^2 + K\right]\varphi_\theta^2. \tag{8.10.11}$$

We are going to demonstrate that, after a slight modification, all terms of this identity are well-defined on the whole of ΩM, i.e., are independent of the choice of isothermic coordinates.

First of all, the operator $H = e^{-\mu}L$ of differentiation with respect to the geodesic flow is well-defined on ΩM. The same is true for the operator $H^\perp = e^{-\mu}L^\perp$ as is seen from the equality $H^\perp = [\frac{\partial}{\partial\theta}, H] + c\frac{\partial}{\partial\theta}$ that follows from (4.4.9).

The 2-form

$$d\sigma = e^{2\mu}dx \wedge dy$$

on ΩM is the pull-back of the Riemannian surface form of M under the projection $\Omega M \to M$. Therefore $d\sigma$ is independent of the choice of isothermic coordinates.

Let us recall that there is the standard contact structure on ΩM with the contact form η that is written in isothermic coordinates as follows:

$$\eta = e^{\mu}(\cos\theta dx + \sin\theta dy). \tag{8.10.12}$$

Therefore the 3-form

$$d\Sigma = -\eta \wedge d\eta = e^{2\mu}dx \wedge dy \wedge d\theta$$

is defined globally on ΩM.

After multiplication by $dx \wedge dy \wedge d\theta$, equality (8.10.11) can be written in the form

$$e^{-2\mu}\left\{[\varphi_x + (\mu_y - e^{\mu}c\sin\theta)\varphi_\theta]^2 + [\varphi_y + (-\mu_x + e^{\mu}c\cos\theta)\varphi_\theta]^2\right\}d\Sigma =$$

$$\left[-4(H^{\perp}\varphi - \frac{1}{2}c\varphi_\theta)^2 + (Hc + 2c^2 + K)\varphi_\theta^2\right]d\Sigma - d\omega, \tag{8.10.13}$$

where

$$\omega = \left[\varphi_x\varphi_\theta + (\mu_y - e^{\mu}c\sin\theta)\varphi_\theta^2\right]dx \wedge d\theta + \left[\varphi_y\varphi_\theta + (-\mu_x + e^{\mu}c\cos\theta)\varphi_\theta^2\right]dy \wedge d\theta +$$

$$+ \left[e^{2\mu}H^{\perp}\varphi \cdot H\varphi + (\mu_x - e^{\mu}c\cos\theta)\varphi_x\varphi_\theta + (\mu_x - e^{\mu}c\sin\theta)\varphi_y\varphi_\theta\right]dx \wedge dy. \tag{8.10.14}$$

We will show that the 2-form ω is independent of the choice of isothermic coordinates and therefore is defined globally on ΩM. To this end we rewrite (8.10.14) in the form

$$\omega = \varphi_\theta(\omega_1 - c\omega_2) + H^{\perp}\varphi \cdot H\varphi \cdot d\sigma,$$

where

$$\omega_1 = (\varphi_x + \mu_y\varphi_\theta)dx \wedge d\theta + (\varphi_y - \mu_x\varphi_\theta)dy \wedge d\theta + (\mu_x\varphi_x + \mu_y\varphi_y)dx \wedge dy \tag{8.10.15}$$

and

$$\omega_2 = e^{\mu}\left[\varphi_\theta \sin\theta dx \wedge d\theta - \varphi_\theta \cos\theta dy \wedge d\theta + (\varphi_x\cos\theta + \varphi_y\sin\theta)dx \wedge dy\right]. \tag{8.10.16}$$

The quantities φ_θ, c and $H^{\perp}\varphi \cdot H\varphi \cdot d\sigma$ are independent of the choice of isothermic coordinates. We have to prove the same for the forms ω_1 and ω_2.

(8.10.12) and (8.10.16) imply the equality

$$\omega_2 = \varphi_\theta d\eta + H\varphi \cdot d\sigma$$

that proves the desired property of ω_2.

Let (x, y) and (x', y') be two systems of isothermic coordinates on M such that the metric is given by (4.4.1) in the first system and by the similar formula

$$ds^2 = e^{2\mu'(x', y')}(dx'^2 + dy'^2)$$

in the second system. In the intersection of their domains, the systems are connected by the transformation formulas

$$x' = x'(x, y), \quad y' = y'(x, y).$$

Since the transform $(x, y) \mapsto (x', y')$ is conformal, the transformation functions satisfy the Cauchy — Riemann equations

$$\alpha = \frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}, \quad \beta = \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x}$$

that imply the relations

$$\alpha_x = -\beta_y, \quad \alpha_y = \beta_x. \tag{8.10.17}$$

The functions μ and μ' are connected by the equality

$$\mu = \mu' + \frac{1}{2}\ln(\alpha^2 + \beta^2) \tag{8.10.18}$$

as is seen from the relations

$$d\sigma = e^{2\mu}dx \wedge dy = e^{2\mu'}dx' \wedge dy'.$$

The corresponding angle coordinates θ and θ' on ΩM are related as follows:

$$\theta' = \theta + \theta'_0(x, y), \quad \theta'_0 = \arctan\left(-\frac{\beta}{\alpha}\right).$$

The latter equality implies that

$$\frac{\partial \theta'_0}{\partial x} = \frac{\beta\alpha_x - \alpha\beta_x}{\alpha^2 + \beta^2}, \quad \frac{\partial \theta'_0}{\partial y} = \frac{\beta\alpha_y - \alpha\beta_y}{\alpha^2 + \beta^2}.$$

We have to prove that the forms

$$\omega_1 = \varphi_x dx \wedge (\mu_x dy + d\theta) + \varphi_y dy \wedge (-\mu_y dx + d\theta) + \varphi_\theta(\mu_y dx - \mu_x dy) \wedge d\theta \qquad (8.10.19)$$

and

$$\omega'_1 = \varphi_{x'} dx' \wedge (\mu'_{x'} dy' + d\theta') + \varphi_{y'} dy' \wedge (-\mu'_{y'} dx' + d\theta') + \varphi_\theta(\mu'_{y'} dx' - \mu'_{x'} dy') \wedge d\theta' \qquad (8.10.20)$$

coincide. Substituting the expressions

$$\varphi_x = \frac{\partial x'}{\partial x}\varphi_{x'} + \frac{\partial y'}{\partial x}\varphi_{y'} + \frac{\partial \theta'_0}{\partial x}\varphi_\theta = \alpha\varphi_{x'} - \beta\varphi_{y'} + \frac{\beta\alpha_x - \alpha\beta_x}{\alpha^2 + \beta^2}\varphi_\theta,$$

$$\varphi_y = \frac{\partial x'}{\partial y}\varphi_{x'} + \frac{\partial y'}{\partial y}\varphi_{y'} + \frac{\partial \theta'_0}{\partial y}\varphi_\theta = \beta\varphi_{x'} + \alpha\varphi_{y'} + \frac{\beta\alpha_y - \alpha\beta_y}{\alpha^2 + \beta^2}\varphi_\theta$$

into (8.10.19), we obtain

$$\omega_1 = \varphi_{x'}\left[\alpha dx \wedge (\mu_x dy + d\theta) + \beta dy \wedge (-\mu_y dx + d\theta)\right] +$$

$$+ \varphi_{y'}\left[-\beta dx \wedge (\mu_x dy + d\theta) + \alpha dy \wedge (-\mu_y dx + d\theta)\right] +$$

$$+ \varphi_\theta\left[\frac{\beta\alpha_x - \alpha\beta_x}{\alpha^2 + \beta^2} dx \wedge (\mu_x dy + d\theta) + \frac{\beta\alpha_y - \alpha\beta_y}{\alpha^2 + \beta^2} dy \wedge (-\mu_y dx + d\theta) + (\mu_y dx - \mu_x dy) \wedge d\theta\right].$$

Comparing the latter formula with (8.10.20), we see that the equality $\omega_1 = \omega'_1$ is equivalent to the following three relations:

$$\alpha dx \wedge (\mu_x dy + d\theta) + \beta dy \wedge (-\mu_y dx + d\theta) = dx' \wedge (\mu'_{x'} dy' + d\theta'), \qquad (8.10.21)$$

$$-\beta dx \wedge (\mu_x dy + d\theta) + \alpha dy \wedge (-\mu_y dx + d\theta) = dy' \wedge (-\mu'_{y'} dx' + d\theta'),$$

$$\frac{\beta\alpha_x - \alpha\beta_x}{\alpha^2 + \beta^2} dx \wedge (\mu_x dy + d\theta) + \frac{\beta\alpha_y - \alpha\beta_y}{\alpha^2 + \beta^2} dy \wedge (-\mu_y dx + d\theta) +$$

$$+ (\mu_y dx - \mu_x dy) \wedge d\theta = (\mu'_{y'} dx' - \mu'_{x'} dy') \wedge d\theta'.$$

We will prove only the first of these equalities because the last two are proved in a similar way.

Inserting the expressions

$$dx' = \alpha dx + \beta dy, \quad dy' = -\beta dx + \alpha dy,$$

$$d\theta' = d\theta + \frac{\beta\alpha_x - \alpha\beta_x}{\alpha^2 + \beta^2} dx + \frac{\beta\alpha_y - \alpha\beta_y}{\alpha^2 + \beta^2} dy$$

into the right-hand side of (8.10.21), we see that this equality is equivalent to the following one:

$$\alpha\mu_x + \beta\mu_y = \alpha\left(\alpha\mu'_{x'} + \frac{\beta\alpha_y - \alpha\beta_y}{\alpha^2 + \beta^2}\right) - \beta\left(-\beta\mu'_{x'} + \frac{\beta\alpha_x - \alpha\beta_x}{\alpha^2 + \beta^2}\right). \qquad (8.10.22)$$

By (8.10.18),

$$\mu_x = \alpha\mu'_{x'} - \beta\mu'_{y'} + \frac{\alpha\alpha_x + \beta\beta_x}{\alpha^2 + \beta^2}, \quad \mu_y = \beta\mu'_{x'} + \alpha\mu'_{y'} + \frac{\alpha\alpha_y + \beta\beta_y}{\alpha^2 + \beta^2}.$$

Inserting these expressions into the left-hand side of (8.10.22), we see that this equality is equivalent to the following one:

$$\alpha(\alpha\alpha_x + \beta\beta_x) + \beta(\alpha\alpha_y + \beta\beta_y) = \alpha(\beta\alpha_y - \alpha\beta_y) - \beta(\beta\alpha_x - \alpha\beta_x)$$

that is valid by (8.10.17). So the independence of the form ω of the choice of isothermic coordinates is proved.

We have thus proved that the right-hand side of equality (8.10.13) is a well-defined form on the whole of ΩM. The same is true for the left-hand side because of the equality. Integrating (8.10.13) over ΩM, we obtain

$$\int\limits_{\Omega M} \left\{ e^{-2\mu} \left[\varphi_x + (\mu_y - e^\mu c \sin\theta)\varphi_\theta \right]^2 + e^{-2\mu} \left[\varphi_y + (-\mu_x + e^\mu c \cos\theta)\varphi_\theta \right]^2 + \right.$$

$$\left. + 4(H^\perp \varphi - \frac{1}{2} c\varphi_\theta)^2 \right\} d\Sigma = \int\limits_{\Omega M} (Hc + 2c^2 + K)\varphi_\theta^2 \, d\Sigma. \tag{8.10.23}$$

Equality (8.10.23) is valid for an arbitrary function $c \in C^\infty(\Omega M)$. Let now $b \in C(\Omega M)$ be the function constructed in Lemma 8.10.2. With the help of Lemma 8.7.1, we can find a sequence of smooth functions $c_k \in C^\infty(\Omega M)$ $(k = 1, 2, \ldots)$ such that c_k and Hc_k converge uniformly on ΩM to b and Hb respectively as $k \to \infty$. Writing down equality (8.10.23) for $c = c_k$ and passing to the limit in the equality as $k \to \infty$, we obtain the same equality (8.10.23) with b in place of c. By Lemma 8.10.1, the right-hand side of the latter equality is nonpositive. Since the integrand on the left-hand side is nonnegative, the equality implies that the integrand is identical zero, i.e., that

$$\varphi_x + (\mu_y - e^\mu b \sin\theta)\varphi_\theta = 0, \quad \varphi_y + (-\mu_x + e^\mu b \cos\theta)\varphi_\theta = 0.$$

In particular,

$$H\varphi = e^\mu \left\{ \cos\theta \left[\varphi_x + (\mu_y - e^\mu b \sin\theta)\varphi_\theta \right] + \sin\theta \left[\varphi_y + (-\mu_x + e^\mu b \cos\theta)\varphi_\theta \right] \right\} = 0.$$

This means that the function φ is constant on every orbit of the geodesic flow. Since there exists such an orbit dense in ΩM, the function $\varphi = u_{\theta\theta} + u$ is constant on ΩM. This means that the function u is representable in the form

$$u(x, y, \theta) = u_0 + u_1(x, y) \cos\theta + u_2(x, y) \sin\theta \tag{8.10.24}$$

in the domain of an isothermic coordinate system, where u_0 is a constant.

The rest of the proof is similar to the end of Section 4.4. Substituting the expression (8.10.24) for u into the kinetic equation (8.10.1), we obtain $f = dv$ for the 1-form $v = e^\mu(u_1 dx + u_2 dy)$. The latter form is easily seen to be well-defined on the whole of M. The theorem is proved.

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